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Causal structures of pp-waves

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Abstract: We discuss the causal structure of pp-wave spacetimes using the ideal point construction outlined by Geroch, Kronheimer, and Penrose. This generalizes the recent work of Marolf and Ross, who considered similar issues for plane wave spacetimes. We address the question regarding the dimension of the causal boundary for certain specific pp-wave backgrounds. In particular, we demonstrate that the pp-wave spacetime which gives rise to the $\mathcal{N} = 2$ sine-Gordon string world-sheet theory is geodesically complete and has a one-dimensional causal boundary.

Keywords: Penrose limit and pp-wave background, Classical Theories of Gravity.

Contents

1. Introduction

pp-waves (or "plane-fronted waves with parallel rays") are all spacetimes with covariantly constant null Killing field. In general relativity, they form simple solutions to Einstein's equations with many curious properties. The presence of the covariantly constant null Killing field implies that these spacetimes have vanishing scalar curvature invariants, much the same as flat space. *Plane waves* are a subset of these which have in addition an extra "planar" symmetry along the wavefronts. They can be thought of as arising from the so-called Penrose limit [1] of any spacetime, which essentially consists of zooming in onto any null geodesic in that spacetime. Nevertheless, these are distinct from flat spacetime and their structure is much richer. Interestingly, as shown by Penrose in [2], plane wave spacetimes are not globally hyperbolic, so that there exists no Cauchy hypersurface from which a causal evolution would cover the entire spacetime. This automatically implies that even the causal structure of pp-waves is different from that of flat spacetime.

pp-wave spacetimes are especially important within the context of string theory. This is because they yield exact classical backgrounds for string theory, since all curvature invariants, and therefore all α' corrections, vanish [3, 4]. Hence the pp-wave spacetimes correspond to exact conformal field theories. Because of this fact, they provide much-needed examples of classical solutions in string theory, which can in turn be used as toy models for studying its structure and properties. Plane waves happen to be even simpler, for the action in light-cone gauge is quadratic.

While this fact has been appreciated for some time [3, 4], only recently have plane waves received significant attention, mainly initiated by the work of Berenstein, Maldacena, and Nastase (BMN) [5], based on the AdS/CFT correspondence [6, 7, 8, 9]. These authors proposed a very interesting solvable model of string theory in Ramond-Ramond backgrounds by taking the Penrose limit of $AdS_5 \times \mathbf{S}^5$ spacetime [5, 10], the holographic dual of $d = 4$, $\mathcal{N} = 4$ Super-Yang-Mills theory. This maximally supersymmetric plane wave solution of type-IIB supergravity [11] (henceforth BMN plane wave) happens to be the simplest example of a sigma model with Ramond-Ramond background that is solvable [12]. Further developments in this area include other interesting solvable or integrable world-sheet theories [13, 14].

A very intriguing aspect of the BMN plane wave is that its conformal boundary happens to be a one-dimensional null line. This was first demonstrated in [15] using the standard technique of conformally mapping the plane wave spacetime into the Einstein Static Universe to construct the Penrose diagram. Later Marolf and Ross [16] showed the same using a more sophisticated technique of adding 'points at infinity', a construction dating back to the work of Geroch, Kronheimer and Penrose [17]. The latter technique has the added advantage of being applicable for spacetimes that are not conformally flat as opposed to the plane wave solution arising from the Penrose limit of $AdS_5 \times \mathbf{S}^5$.

In a preceding paper [18], we had asked a general question: *Do pp-waves admit event horizons?* The primary motivation for the same was to check whether there were black hole like spacetimes admitting a covariantly constant null Killing field. If the answer to the question were in the affirmative, we would have black hole solutions that remain exact conformal field theories to all orders in the perturbative α' expansion. In addition, one might hope to be able to delve deeper into the mysteries of black holes using perturbative string technology, provided that these solutions proved amenable to light-cone quantization.

In [18] we have argued that plane waves cannot admit event horizons, because every point of the spacetime can communicate "out to infinity". Since black holes are defined as the regions bounded by event horizons, this automatically shows that there can't be black hole pp-waves.

While the fact that the spacetime admits no event horizons gives us an important information about its causal structure, it of course does not determine this causal structure in its entirety. One may well ask, why should we want to know this causal structure of a spacetime? The motivation for such analysis, apart from its obvious interest to general relativity, is that the causal structure of a spacetime gives us some important information about the spacetime. In particular, in the spirit of AdS/CFT correspondence, the structure of the boundary may hint at possible background on which a potential dual theory would live. While it is certainly not guaranteed that there will be a dual theory "living on the boundary" of a spacetime, knowledge of the causal structure might prove of some use in determining the same. Also knowledge of the causal structure enables one to make stronger arguments than those presented in [18] regarding the question of the presence/absence of event horizons. By outlining the general properties of the causal structure that pp-wave spacetimes satisfy, we put our analysis at a level of greater robustness.

An important issue that the knowledge of the causal structure allows us to to discuss is the dimensionality of the causal boundary. Most typical d-dimensional spacetimes have a $d-1$-dimensional boundary. Such is true for all the standard examples, such as Minkowski, Anti-deSitter and deSitter spacetimes, and for more generic solutions which asymptote to the same, such as black holes in these spacetimes. The startling fact which is revealed by the analysis of [15, 16] is that for the maximally supersymmetric plane wave solution of type-IIB supergravity the boundary is one-dimensional! In fact, the same is true for certain other classes of plane waves such as those arising from the Penrose limits of $AdS_7 \times \mathbf{S}^4$, $AdS_4 \times \mathbf{S}^7$ and the near horizon geometry of D4 branes [16], among others. There however, are examples of plane wave spacetimes where the boundary isn't one-dimensional. We wish to ask whether the pp-wave spacetimes share similar properties and will argue for some classes of pp-waves (including some that lead to integrable world-sheet theories) that the causal boundary[1] is indeed one-dimensional.

The outline of this paper is as follows. In the following short section, we review certain basic aspects of plane wave and pp-wave spacetimes, mainly with the view of setting up notation. In section 3, we review the work of Geroch, Kronheimer and Penrose, providing the ingredients necessary for determining the causal structure of any spacetime. We then turn to the question of causal structure of general plane waves in section 4, reviewing the arguments of Marolf and Ross, and comment on some generalizations, and present a few examples. In section 5, we turn to discussing the causal structure of general pp-waves and construct the causal structure for certain interesting pp-wave solutions. We end in section 6 with a brief summary and more general discussion of singularities and causal structure. In appendix A we collect some useful facts about plane and pp-wave spacetimes, and in appendix B we present details relating to null geodesics in particular vacuum pp-waves.

2. Notation and terminology

To set the notation and re-emphasize terminology, we will write explicitly three classes of spacetimes, in decreasing generality. The *pp-wave* spacetimes, which are defined as all spacetimes admitting a covariantly constant null Killing field, can be written as

$$ds^2 = -2\,du\,dv - F(u,x^i)\,du^2 + A_i(u,x^i)\,du\,dx^i + g_{ij}(x^i)\,dx^i\,dx^j\,. \tag{2.1}$$

We shall in the following be working with spacetimes wherein $A(u,x^i) = 0$ in order to maintain the simplicity of the discussion. For the case of pure gravity, vacuum Einstein's

[1] There is a slight subtlety relating to the distinction between causal boundary and conformal boundary, having to do with the topology of the completed manifold. While we mention this at the end of section 3, in the present work we bypass these topological subtleties by confining our discussion to causal structures, as was done by [16].

equations dictate that $F(u, x^i)$ satisfy the transverse Laplace equation for each u and that the transverse space be Ricci flat. $F(u, x^i)$, however, can be an arbitrary function of u. Another simplification that we will make is to consider pp-wave spacetimes with flat transverse part, i.e., we will consider spacetimes with the metric

$$ds^2 = -2\,du\,dv - F(u, x^i)\,du^2 + dx^i\,dx^i\,. \tag{2.2}$$

Plane wave spacetimes are those where the harmonic function in (2.2) is in fact quadratic, $F(u, x^i) = f_{ij}(u)\,x^i x^j$, so that plane waves can be written as

$$ds^2 = -2\,du\,dv - f_{ij}(u)\,x^i x^j\,du^2 + dx^i\,dx^i\,. \tag{2.3}$$

Here, $f_{ij}(u)$ can be any function of u, subject to the constraint that for each u, f_{ij} is symmetric and traceless (the latter being required by vacuum Einstein's equations). As suggested by the name, these metrics have an extra "plane" symmetry, which contains the translations along the wave-fronts in the transverse directions. This can be seen explicitly by casting (2.3) into the Rosen form,[2]

$$ds^2 = -2\,dU\,dV + C_{ij}(U)\,dX^i\,dX^j\,. \tag{2.4}$$

The *homogeneous plane waves* further specialize (2.3) by taking out f's dependence on u,

$$ds^2 = -2\,du\,dv - f_{ij}\,x^i x^j\,du^2 + dx^i\,dx^i\,. \tag{2.5}$$

The BMN plane wave metric [5], found earlier by [11], belongs to this last class, for the special case $f_{ij} = \mu^2\,\delta_{ij}$, and $u \equiv x^+, v \equiv x^-$ in their notation. In fact, in the constant f case, we can diagonalize the metric completely, which leads to substantial simplification in the analysis, as used e.g. by [16].

All the aforementioned spacetimes have a covariantly constant null Killing vector, given by $p^a = (\partial/\partial v)^a$. The fact that this is a null Killing vector is obvious from the metric, while its being covariantly constant may be inferred from the vanishing of the Christoffel symbols $\Gamma^v{}_{\mu v}$.

3. Causal structure generalities

The most conventional, and often the easiest, way to determine the causal structure of a spacetime is to conformally map the spacetime into the Einstein Static Universe (ESU), and see where the conformal factor diverges. This method was employed e.g. by [15] to determine the asymptotic structure of the BMN plane wave. This approach, however, only works for a limited class of spacetimes, for which a conformal factor exists, i.e., conformally flat ones (defined by the vanishing of the Weyl tensor). As pointed out by [16],

[2]Typically, this metric is not geodesically complete because of coordinate singularities, but the Brinkman form (2.3) does cover the full spacetime. The coordinate transformation from one form into the other is given e.g. in [19]. For metric of the Brinkman form $ds^2 = -2dudv - f(u)x^2du^2 + dx^2$, the coordinate transformation $\{u = U, x = h(U)X, v = V + \frac{1}{2}h(U)h'(U)X^2\}$ where $h(U)$ satisfies $h''(U) + f(U)\,h(U) = 0$, casts this metric into the Rosen form $ds^2 = -2\,dU\,dV + h(U)^2\,dX^2$.

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this condition is not satisfied by general plane waves (nor by the more general pp-waves). In fact, the Weyl tensor for the metric (2.2) in d dimensional spacetime is given by

$$C_{uiuj} = \frac{1}{2}\left(\partial_i \partial_j F(u,x) - \frac{1}{d-2}\,\delta_{ij}\,\sum_k \partial_k^2 F(u,x)\right) \tag{3.1}$$

where $i,j = 1,\ldots,d-2$, which, in the plane wave spacetimes (2.3) reduces to

$$C_{uiuj} = f_{ij}(u) - \frac{1}{d-2}\,\delta_{ij}\,\sum_k f_{kk}(u)\,. \tag{3.2}$$

Since the only nonzero component of the Ricci tensor is

$$R_{uu} = \frac{1}{2}\nabla_T^2\,F(u,x^i) \tag{3.3}$$

and the Ricci scalar then automatically vanishes, the only vacuum pp-wave which is conformally flat is the trivial (flat spacetime) one, where $F(u,x^i) = a + b_i\,x^i$, which can be cast into the form (2.2) with $F(u,x^i) \equiv 0$ by appropriate coordinate transformation. On the other hand, certain nontrivial plane (or pp) waves can be conformally flat if we allow fluxes, such as in the BMN plane wave, where $F(u,x^i) = \mu^2 \sum_i (x^i)^2$. For this class of spacetimes one may use the ESU conformal mapping procedure to determine the causal structure. Unfortunately, this class of spacetimes is rather limited; for instance, the Penrose limits of $AdS_4 \times \mathbf{S}^7$ and $AdS_7 \times \mathbf{S}^4$ analogous to the BMN plane wave do *not* fall into this category. Similar considerations hold for the case of general pp-waves.

To bypass that obstacle, [16] used a more direct approach to find the causal structure of these spacetimes, based on the method introduced by Geroch, Kronheimer, and Penrose [17]. In the present section, we will present a self-contained review of the method of ideal point construction.

3.1 Review of ideal point construction

To find the causal structure of a given spacetime, [17] complete the spacetime by "ideal points", corresponding, roughly speaking, to the endpoints of inextendible causal curves γ. This procedure of adding the boundary to our spacetime allows us to apply the usual notions of causality to this boundary, thereby extracting the causal structure. We will first develop the necessary terminology and then outline the prescription in more detail.

Let $I^-(p)$ denote the indecomposable past-set (IP) of a given point p in the spacetime. This is basically the collection of points from which there exists a future-directed causal curve[3] to p, or in other words, the points lying in the past light-cone of p. We can consider instead of a point a future-directed causal curve itself, and denote by $I^-[\gamma]$ the IP associated with the curve γ. Of course, this is simply the union of the IPs associated with all the points lying on the curve γ, i.e., $I^-[\gamma] = \bigcup_i I^-(p_i)$ for all $p_i \in \gamma$. Now an IP is a proper IP

[3] We define *causal* curves to be timelike or null, so that unlike the usual convention, we take the set $I^-(p)$ to be closed. This makes our notation and arguments cleaner; however, we could have taken the stricter definition of IPs defined by timelike curves only, and used strict inequalities correspondingly.

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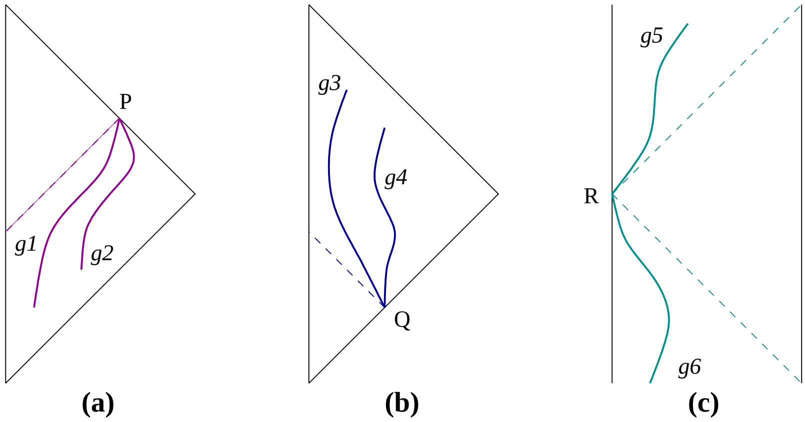

Figure 1: Example of identifications between TIPs and TIFs. (a) The causal curves $g1$ and $g2$ have the same TIP; (b) Similarly, the causal curves $g3$ and $g4$ have the same TIF; (c) Finally, the TIF of $g5$ and TIP of $g6$ are identified. In each case, there is correspondingly a single ideal point (labeled P, Q, and R, respectively) attached.

or PIP if there exists some point p in the manifold such that $I^-[\gamma] = I^-(p)$; else the IP is a terminal IP or TIP. TIPs can be seen to correspond to future endless causal curves. We can extend the definitions to past-directed causal curves, thereby creating indecomposable future-set (IF) and thence PIF and TIF respectively. Note that the curves can be endless either because they run off to infinity, or because they hit a physical singularity (since singularities are by definition not a part of our physical spacetime). Hence the set of ideal points will describe both the causal boundary of our spacetime, as well any singularities.

To construct the causal structure of a spacetime, the prescription is as follows:

1. First find the set of TIP(F)s, or "terminal indecomposable past (future) sets", $I^{\pm}[\gamma]$, of all causal curves γ. Since curves corresponding to TIP(F)s are future (past) endless, we can add an "ideal point" to our physical spacetime. This is a basically a point at infinity, such that the TIPs of the curves would be converted into PIPs were this ideal point part of the spacetime manifold. The ideal points are thus added in by hand to the original manifold and one has two completions of the original manifold M, $\hat{M}$ and $\check{M}$ corresponding to the original manifold with all the ideal points given by TIPs and TIFs respectively. This, however, is not the entire story: whereas we have added points on which causal curves end, we haven't yet specified the relations between these added ideal points, which brings us to the second element of the prescription.

2. Consider the manifold $M^\sharp = \hat{M} \cup \check{M}$. The global completion of M to a manifold with a nice conformal boundary is defined to be $\bar{M} = M^\sharp/\Gamma$ where Γ is the minimal set of identifications between the ideal points thus added, such that $\bar{M}$ is a smooth Hausdorff manifold.

The second point, concerning identifications, deserves a bit more explanation. Since the ideal points were defined through the TIP(F)s, distinct ideal points should correspondingly have distinct TIP(F)s. In other words, different causal curves may nevertheless have identical past and/or future, so that their TIPs and/or TIFs are identical sets; if such is the case, the corresponding ideal points are identified. This is illustrated by the examples in figure 1: In figure 1a, the causal curves $g1$ and $g2$ have the same past, lying below the dashed line. Correspondingly, we attach only a single ideal point, P, to their TIP. Similarly, in figure 1b, there is only a single ideal point Q corresponding to the TIF of the causal curves $g3$ and $g4$. A more subtle (but very important) identification is illustrated in figure 1c, where the past endpoint of curve $g5$, R, is the future endpoint of a curve $g6$. Thus, we see that in this case, some TIPs may need to be identified with some TIFs. This necessarily occurs whenever the boundary is timelike, such as for AdS; but we will see below that it can in fact occur for null boundaries as well.

Before proceeding, we should put in a side cautionary remark. An important feature associated with the identifications of the ideal points has to do with the topology of the resulting completed spacetime. One expects on physical grounds that conformally completed spacetime manifold has a Hausdorff topology, enabling one to distinguish between distinct points. If one uses the GKP construction to causally complete the spacetime with ideal points, the completed manifold is by definition a Hausdorff spacetime; so one might expect that the conformal completion of the spacetime should also have the requisite Hausdorff topology. However, this may be in general a more subtle issue than the simple set of identifications we discuss above, i.e. based on the causal properties associated with the ideal points [20]. In particular, the ideal points added as endpoints to causal curves may not correspond to endpoints of spacelike curves, and vice-versa. Discussion of these issues and corresponding improvement of the GKP scheme is to appear in [21]. In what follows, we shall ignore this subtlety, and instead concentrate on identifications between the ideal points ensuing from the requirement of causality. This allows us to talk only of the causal boundary of the spacetime, not the conformal boundary as one is usually accustomed to.

4. Causal structure of generic plane wave spacetimes

We now proceed to use the Geroch-Kronheimer-Penrose [17] method to ascertain the causal structure properties of plane wave backgrounds. The essential ingredients for the construction as we have discussed earlier are the knowledge of the TIP(F)s in the spacetime, and the identifications between them. This was done by [16] for the homogeneous plane waves (2.5) in detail and discussed for a few more general cases. Here we review the method used in [16] (with some modifications), and then extend it to find the causal structure of more general plane waves.

The generic plane wave has an arbitrary functional dependence on the coordinate u, through the function matrix $f_{ij}(u)$ appearing in (2.3). One would presume that, without knowledge about some characteristic features of the matrix $f_{ij}(u)$, making detailed statements regarding the causal structure of the spacetime would be well nigh impossible. However, one can extract a lot of information about the causal structure without knowing the details of the functional form, simply by resorting to local analysis. In particular, we will be able to use the fact that $f_{ij}(u)$ is a real, symmetric matrix, with its elements being real continuous functions, to first approximate the functional form of $f_{ij}(u)$ by constants

$f_{ij}(u_0)$ in a neighbourhood of $u = u_0$, implying that in this neighbourhood the metric is of the homogeneous plane wave form (2.5). Having done so, we can perform a rotation in the transverse space to put the metric in the form

$$ds^2 = -2\, du\, dv - f_i^{(0)}\, (x^i)^2\, du^2 + dx^i\, dx^i \tag{4.1}$$

with $f_i^{(0)}$ being the eigenvalues of the matrix $f_{ij}(u_0)$. The only requirement we will have, stemming from the energy conditions, is that $\mathrm{Tr}(f_{ij}(u)) \geq 0$ for all u.

In the next subsection, we turn to the question of constructing the TIPs for a general plane-wave. In terms of the coordinates used in (2.3), a future-endless curve can either end at infinite u or at finite u (in which case some other coordinate diverges). We first turn to this latter case.

4.1 Review of TIPs of the general plane wave

We will now rephrase the proof, used by Marolf and Ross in the appendix of [16], of the claim that the TIP of any causal curve γ which asymptotes in the future to a finite $u = u_1$ is given by the set of all points with $u \leq u_1$. We write this claim in a somewhat condensed notation as

$$\mathrm{TIP}\,[\gamma(u \to u_1)] = \{(u, v, x^i) : u \leq u_1\}\,. \tag{4.2}$$

Since u is just a parameter along the curve, we can translate it by $u \to u + u_1$, so that (4.2) is equivalent to the claim that $\mathrm{TIP}\,[\gamma(u \to 0)] = \{(u, v, x^i) : u \leq 0\}$. To prove this, it in fact suffices to show that the TIP of any curve γ asymptoting to $u = 0$ contains all points on the surface $u = -\delta$ for arbitrarily small δ,

$$\forall\, \delta > 0\,, \qquad \mathrm{TIP}\,[\gamma(u \to 0)] \supset \{(u, v, x^i) : u = -\delta\}\,. \tag{4.3}$$

The rest follows by the following series of steps:

- Any point with $u < -\delta$ is in the past of some point in the $u = -\delta$ surface, so (4.3) implies that $\forall\, \delta > 0$, $\mathrm{TIP}\,[\gamma(u \to 0)] \supset \{(u, v, x^i) : u \leq -\delta\}$.

- No point with $u > 0$ can be in the TIP of any causal curve γ which asymptotes to $u = 0$, because u can only increase along (future-directed) causal curves. This can be seen as follows: Take any causal curve γ and any point p_0 on γ. We can use the plane symmetry to translate the x^is to the origin along the constant u plane. Now at p_0, the causal relation (in the translated coordinates) then becomes $-2\,\dot{u}\,\dot{v} + (\dot{x^i})^2 \leq 0$, which is exactly the same relation we would obtain in flat space. But in the flat space, it is clear that u cannot decrease along any causal curve (since if it did, so would v in order to maintain causality, but then the curve would by definition be past-directed). This shows that $\dot{u} \geq 0$ at p_0, but since p_0 was arbitrary, we have shown that u can only increase along future-directed causal curves.

- Finally, TIP is by definition a closed set, so that taking $\delta \to 0$ leads to the desired statement that $\mathrm{TIP}\,[\gamma(u \to 0)] = \{(u, v, x^i) : u \leq 0\}$.

Thus, if we can prove the claim (4.3), we are done.

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To simplify the proof even further, we now make use of the transverse-plane symmetry of plane waves (which is not present for generic pp-waves): we can translate the point $(u = -\delta, v, x^i)$ along the wavefront to $(u = -\delta\,,\, v = 0\,,\, x^i = 0)$. We would of course need to apply the same translation to our "set of all causal curves $\gamma(u \to 0)$", but this set is by definition invariant. Thus, to prove (4.3), it suffices to show that the point $(u = -\delta\,,\, v = 0\,,\, x^i = 0)$ is in the past of *any* causal curve which asymptotes to $u = 0$,

$$\forall\ \delta > 0\,, \qquad I^-\left[\gamma(u \to 0)\right] \ni (u = -\delta\,,\, v = 0\,,\, x^i = 0) \tag{4.4}$$

But by definition of $I^\pm$, for any two points p and q, $q \in I^-(p)$ iff $p \in I^+(q)$. This means that (4.4) is equivalent to the claim

$$\forall\ \delta > 0\,, \quad \exists \quad \varepsilon > 0 \quad \text{s.t.} \quad \gamma(u > -\varepsilon) \subset I^+(u = -\delta\,,\, v = 0\,,\, x^i = 0) \tag{4.5}$$

i.e., any causal curve γ which asymptotes to $u = 0$ must enter the future of the point $(u = -\delta\,,\, v = 0\,,\, x^i = 0)$.

Now, to show this, it of course suffices to prove that any causal $\gamma(u \to 0)$ enters into any subset of $I^+(u = -\delta\,,\, v = 0\,,\, x^i = 0)$; in particular that it enters into the region bounded by null (and therefore causal) curves $\{\mathcal{C}\}$ emerging from $(u = -\delta\,,\, v = 0\,,\, x^i = 0)$. Note that if these null curves $\{\mathcal{C}\}$ are actually null geodesics, this region is the full[4] $I^+(u = -\delta\,,\, v = 0\,,\, x^i = 0)$; however, it may turn out more convenient not to require that $\{\mathcal{C}\}$ be geodesics. In fact, [16] make such a non-geodesic choice to complete their proof.

Specifically, [16] first choose convenient null curves $\{\mathcal{C}\}$ (see their eq.(A.1)) and construct the region $R_\delta \subset I^+(u = -\delta\,,\, v = 0\,,\, x^i = 0)$ bounded by these curves. Far along the curves, this is given by

$$x^2 \leq 2(1 - \varepsilon_1)\,\delta\,v \tag{4.6}$$

for some ε_1, where $x^2 \equiv \sum_i x^i\,x^i$. Thus, to prove (4.5), we have to show that the coordinates along any causal curve γ satisfy the relation (4.6) sufficiently far along γ (i.e., for u sufficiently close to zero). Recall that for any future-endless γ asymptoting to a finite u plane, at least one other coordinate must diverge along γ; in fact, v must diverge, since by causality v grows faster than any x^i. Now, if $v \to \infty$ as $u \to 0$ while x stays bounded, equation (4.6) is automatically satisfied, so we are done. Therefore we will assume that x also diverges as $u \to 0$, i.e., γ reaches arbitrarily large values of x and v. To bound how fast x can diverge relative to v, [16] use the metric to argue that as $u \to 0$,

$$2\dot{v} \geq (1 - \varepsilon_2)\,\dot{x}^2 \tag{4.7}$$

along any causal curve γ for arbitrarily small ε_2. The causal relation (4.7) can be reproduced by a fiducial metric

$$ds^2_{\text{fid}} = -2\,du\,dv + (1 - \varepsilon_2)\,dx^i\,dx^i \tag{4.8}$$

which gives the finite-difference relation[5] $(\Delta x)^2 \leq \frac{2}{1-\varepsilon_2}\,\Delta v\,\Delta u$. Since $x, v \to \infty$, $\Delta x \to x$ and $\Delta v \to v$, so the relation becomes $x^2 \leq \frac{2}{1-\varepsilon_2}\,v\,\Delta u$. But for $\Delta u \equiv \varepsilon \ll \delta$, we can easily

[4]This is actually true only locally, upto where the geodesics caustic.

[5]This works because (4.8) is flat — otherwise we would have to integrate ds to obtain the correct Δs^2.

arrange for $x^2 \leq 2(1-\varepsilon_1)\,\delta\,v$, which is exactly the relation (4.6) describing the set contained in $I^+(u=-\delta\,,\,v=0\,,\,x^i=0)$. Hence, given any (arbitrarily small) δ, we have shown that for u sufficiently close to 0, $\gamma(u\to 0)$ enters $I^+(u=-\delta\,,\,v=0\,,\,x^i=0)$. **(QED)**

4.2 Comments on nonexistence of horizons

Let us at this point take a detour from constructing the full causal structure of plane waves, and comment on the implications of the results thus far, concerning the nonexistence of event horizons as discussed in [18]. In particular, knowledge of the TIPs will immediately tell us that the spacetimes which are of the plane wave form can not admit event horizons.

Recall that a necessary condition for the existence of an event horizon is that there exist points in the spacetime which are causally disconnected from future infinity, $i^+ \cup \mathcal{I}^+$. Hence, to prove the absence of horizons, it suffices to prove that all points in the spacetime are contained in the past of infinity. By proving that for any finite value of u_1, all points with $u \leq u_1$ are in the past of the ideal point corresponding to the TIP of any causal curve γ which asymptotes to $u = u_1$, we have shown that all of spacetime is causally connected to infinity. This in turn means that there can't be any black holes of the plane wave form, i.e., that plane waves don't admit event horizons.

There is however one subtlety, which we now address. The argument of [16] applies to general plane wave metrics (2.3), with the assumption that $f_{ij}(u)$ is regular in open neighbourhood of $u = u_1$. Now, since $f_{ij}(u)$ is not required to remain regular in a general plane wave, let us classify plane waves into two classes: (1) nonsingular, with $f_{ij}(u)$ remaining finite for all values of u, and (2) singular, where some $f_{ij}(u) \to \infty$ as $u \to u_\infty < \infty$. For nonsingular plane waves, we can consider any point $p_0 = (u_0, v_0, x_0^i)$ of the spacetime, and apply the above proof for $u_1 > u_0$, to show that $p_0 \in \mathrm{TIP}\,[\gamma(u \to u_1)]$, or $p_0 \in I^-\,[\mathcal{I}^+]$. For the second class, with the spacetime becoming singular at some finite value of $u \to u_\infty$, we can apply essentially the same argument as above: The spacetime manifold is an open set (which in particular does not by definition include the singularities), so for any point $p_0 = (u_0, v_0, x_0^i)$ in the spacetime, there exists u_1 such that $u_0 < u_1 < u_\infty$. Applying the proof of [16] to u_1, we again see that $p_0 \in \mathrm{TIP}\,[\gamma(u \to u_1)]$, i.e., p_0 is visible from infinity.

4.3 Causal structure of plane waves

Let us now return to the more general discussion of the causal structure of plane waves. The construction outlined in subsection 4.1, in particular the claim (4.2), tells us that the TIPs are parameterised by a single parameter u.[6] By exchanging past and future, the same conclusion will apply to the TIFs. To complete the discussion of the causal structure, we need to know what are all the identifications between the TIPs and the TIFs. We will first review the simple case of maximally symmetric homogeneous plane wave addressed by [16], and then extend this to more general plane waves.

We will start with the simplest case of the maximally supersymmetric plane wave solution to type-IIB supergravity, given by the metric (2.3) with $f_{ij}(u) = \mu^2\delta_{ij}$. It is

[6] So far, we have said nothing about the TIPs of causal curves which reach infinite u. However, since this is closely tied to the discussion of identifications, we shall first address those, and return to this point afterwards.

shown in [16] that for every u_0, the TIP described by $\{(u, v, x^i) : u \leq u_0\}$ is identified with the TIF $\{(u, v, x^i) : u \geq u_0 + \frac{\pi}{\mu}\}$. In other words, the ideal points corresponding to, heuristically, $(u = u_0, v \to +\infty, x^i)$ and $(u = u_0 + \frac{\pi}{\mu}, v \to -\infty, x^i)$ are identified. This identification rests on the simple fact that the future of every point on the $u = u_0$ plane contains the future of all points on the $u = u_0 + \frac{\pi}{\mu}$ plane. To see this, it suffices to show that there exists a sequence of null geodesics emanating from any point on the $u = u_0$ plane, i.e., null geodesics starting from $p_0 = (u_0, v, x^i)$ for arbitrary v, x^i, which have an accumulation curve ending up at $p_1 = (u_0 + \frac{\pi}{\mu}, -\infty, x^i)$ for some finite but arbitrary x^i. Furthermore, the same is not true for any smaller value of u in p_1. The existence of this accumulation curve for the sequence of null geodesics implies that we can causally reach large negative values of v in finite u. By construction of the sequence, we know that p_1 is in the causal future of p_0, so all points with $u \geq u_0 + \frac{\pi}{\mu}$ are in the causal future of p_0 as well. This is the reason why we need to identify the TIPs and the TIFs: the associated ideal points share the same causal past and future.

This identification has a very important consequence. In particular, taking $u_1 \to \infty$ in the above argument, we see that any causal curve which reaches infinite u has in its past the entire spacetime, so that the whole region of the causal boundary corresponding to infinite u is actually represented by a single ideal point, i^+. Thus, having identified the TIPs and the TIFs we see that the causal boundary of the spacetime is parameterized by a single parameter u, i.e., it is described by a single line! As we mentioned in section 3, the construction of the ideal points allows us to discuss the causal properties of the boundary in the completed spacetime. In particular, as was already demonstrated by [16], this curve of ideal points is locally null.[7]

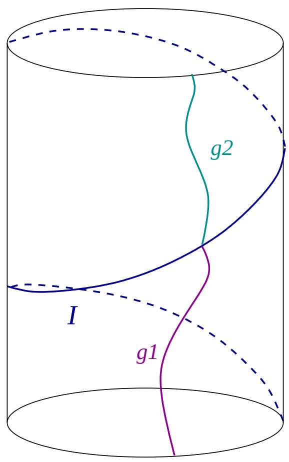

Figure 2: Causal boundary of a plane wave, $\mathcal{I}$: In most cases it is a one-dimensional null line, with some TIPs and TIFs identified, as exemplified by the causal curves $g1$ and $g2$.

This is the first example, promised above, of a null boundary which nevertheless has its ideal points corresponding to both TIPs and TIFs. This point is illustrated in figure 2. The spacetime is conformally mapped to the cylinder (in this particular case, representing the Einstein Static Universe), and its causal boundary $\mathcal{I}$ then winds around this cylinder in a null fashion as shown. Causal curves, such as $g1$, can end on this boundary; but they can also start on it, as $g2$ in figure 2. Note, however, that geodesic observers cannot "pass through" this boundary, because it takes them an infinite proper time to reach it, as guaranteed by the fact that the boundary is non-singular. (This will be contrasted later, for a certain class of pp-waves.) As we will see below, this "null line" nature of the boundary appears much more generically than just for this conformally flat homogeneous plane wave.

[7]Since the identifications cause this line boundary to "wind around" (as can be easily checked by conformally mapping the spacetime into the ESU and seeing the corresponding boundary wind around the cylinder), the ideal points separated by $u > \frac{\pi}{\mu}$ are in fact timelike-separated.

In the case of the generic plane wave, determination of the identifications between the TIPs and TIFs is generally more complicated, since the geodesic equations depend now on the explicit form of the functions $f_{ij}(u)$. Specifically, as derived in appendix A, the geodesic equations are given by

$$\ddot{x}^i + \sum_j f_{ij}(u)\, x^j = 0 \tag{4.9}$$

and $v(u)$ is determined simply from integrating the first order constraint equation,

$$v = \frac{1}{2} \sum_i x^i\, \dot{x}^i + v_0 \tag{4.10}$$

where we can take $\dot{} \equiv d/du$, and v_0 is an arbitrary integration constant which is fixed by the initial conditions.

To see if there are any identifications and to claim that the causal boundary is one-dimensional, it however suffices to show that, starting from any $p_0 = (u_0, v, x^i)$, there is a sequence of null geodesics which accumulate toward large negative values of v whilst keeping the transverse coordinates finite. This will in particular happen so long as the behaviour of the null geodesics continues to be oscillatory as a function of u, for arbitrarily large values of u. Here we are assuming that the functions $f_{ij}(u)$ are regular functions of u. In the case that they are not, we will encounter singularities at the null planes where $f_{ij}(u)$ diverges, as discussed above.

Before we embark on explicitly demonstrating that this is possible for a wide variety of examples, let us for a moment pause to look at the situation in the BMN plane wave, with $f_{ij}(u) = \mu^2 \delta_{ij}$. Here, the geodesic equations (4.9), (4.10) can be solved to give

$$\begin{aligned} x^i(u) &= \frac{\sqrt{2\kappa}}{\mu} \sin(\mu u) \\ v(u) &= \frac{\kappa}{2\mu} \sin(2\mu u) \end{aligned} \tag{4.11}$$

where we have chosen to look at null geodesics emanating from the origin, with velocity $\dot{x}^i(0) = \sqrt{2\kappa}\ \forall\ i$. In the neighbourhood of $u \to \left(\frac{\pi}{\mu}\right)^-$ we see that $x^i(u) \to 0^+$ with $\dot{x}^i(u) < 0$, implying $v(u) < 0$. Now with a suitable choice of the parameter κ one can ensure that $v(u \to \frac{\pi}{\mu}) \to -\infty$, whilst retaining finite values of $x^i(u = \frac{\pi}{\mu})$. For instance, consider a sequence of null geodesics labeled by $\kappa_n = \frac{1}{2} n^{3/2}$. Considering the sequence of points on the aforementioned null geodesics at $u_n = \frac{\pi}{\mu} - \frac{1}{n}$, in the limit $n \to \infty$, we find that these have an accumulation point, $(u = \frac{\pi}{\mu}, v = -\infty, x^i = 0)$. Thus, for the existence of a sequence of null geodesics that accumulate towards large negative values of v, with finite values of transverse coordinates, it suffices to show that the geodesic equations (4.9) admit solutions wherein the curves $x^i(u)$ have zeros at arbitrarily large values of u, with a negative slope. Given that v scales quadratically with the transverse coordinates, it is possible to choose initial conditions such that v gets arbitrarily large and negative. If this behaviour persists for arbitrarily large values of u, then the identifications between the TIPs and the TIFs ensure that the causal boundary is one-dimensional.

It is easy to see that this accumulation point criterion will be satisfied for $f_{ij}(u)$ which are polynomials, trigonometric, or hyperbolic functions. It is possible to make explicit statements if we assume that the matrix $f_{ij}(u)$ is of the form $f_{xj}(u) = 0$ for $j \neq x$ and let $f_{xx}(u) = f(u)$ for some transverse space coordinate x. This simplification will allow us to analyze the geodesic equation (4.9) without concerning ourselves with having to solve a system of coupled oscillators. Let us now try to understand the behaviour of the geodesics for particular cases of the function $f(u)$. Our notation of special functions conforms to the standards of [22].

1. $f(u)$ is a polynomial function of u, say $f(u) = u^n$. In this case we see that the solution to the geodesic equation $\ddot{x}(u) + u^n\, x(u) = 0$ is given in terms of Bessel functions, $x(u) = \sqrt{u}\, J_\nu(\frac{2}{n+2}\, u^{\frac{n+2}{2}})$, with $\nu = \pm\frac{1}{2+n}$. These solutions are clearly oscillatory for arbitrarily large u, given that $\sqrt{u}\, J_\nu(u) \to \cos(u - \frac{\pi}{2}\nu - \frac{\pi}{4})$ for $u \gg 1$.

2. $f(u)$ is an exponential function, say $f(u) = e^u$. Again the solutions are Bessel functions with $x(u) = J_0(2e^{\frac{u}{2}})$ or $x(u) = N_0(2e^{\frac{u}{2}})$, which exhibit oscillatory behaviour for arbitrarily large u.

3. Consider $f(u) = \cos(u)$. The solutions to (4.9) are $x(u) = \mathrm{ce}_0(x, -2)$ or $x(u) = \mathrm{se}_0(x, -2)$, where $\mathrm{ce}_a(x, q)$ and $\mathrm{se}_a(x, q)$ are the periodic Mathieu functions. In this case also we have an oscillatory behaviour.

4. An interesting example to consider is one where $f(u) \to 0$ for large u. One would imagine that in this case, with rapid approach to flat space, the status of the identifications between the TIPs and the TIFs would be problematic. Let us for concreteness consider $f(u) = \frac{1}{1+u^2}$. For this example we can show that $x(u) = \mathrm{F}\left(-\frac{1}{2}(-1)^{1/3}, \frac{1}{2}(-1)^{2/3}; \frac{1}{2}; -u^2\right)$ or $x(u) = u\,\mathrm{F}\left(\frac{1}{2} - \frac{1}{2}(-1)^{1/3}, \frac{1}{2} + \frac{1}{2}(-1)^{2/3}; \frac{3}{2}; -u^2\right)$, where $\mathrm{F}(a, b; c; x)$ denotes the hypergeometric function. Here too it is easy given the explicit solution to convince oneself that there exist identifications.

5. Suppose, on the other hand the approach to flat space is exponential, i.e., $f(u) = e^{-u^2}$. In this case the geodesics are oscillatory in a small neighbourhood of $u = 0$. This oscillatory behaviour ceases for some finite u. While there exist identifications for finite values of u, the structure at large u is akin to flat space. Thus, the causal structure of this spacetime is similar to the sandwich plane wave as discussed in [16].

6. Let us turn to a slightly different behaviour of $f(u)$, one wherein we have a singularity. Consider for instance, $f(u) = A/u^2$. This is the generic behaviour of plane waves limits of black hole spacetimes, when we consider the null geodesics which terminate at the singularity. For this form of $f(u)$, it is easy to see that there are oscillatory solutions as long as $A > 1/4$.

In the preceding discussion we have investigated certain generic classes of functional behaviour for which it is possible to show explicitly where there are identifications between TIPs and TIFs. Let us now see what happens for plane wave spacetimes which can be obtained as Penrose limits of some geometries which are interesting from a string theory point of view. Certain Penrose limits of D-branes were discussed in [16]; here we propose to consider the plane wave limits of spacetimes which holographically encode dynamics of non-local theories, such as little string theory or non-commutative gauge theory. The Penrose limits for these spacetimes were discussed in [23].

NL1. The Nappi-Witten geometry [24],

$$ds^2 = -2\,du\,dv - \ell^2\,\vec{z}^2\,du^2 + d\vec{z}^2 + \sum_{i=1}^{6}(dy^i)^2 \tag{4.12}$$

is obtained as the Penrose limit of the near horizon geometry of NS5-branes [25, 26, 23], by considering null geodesics which have some angular momentum ℓ along the $\mathbf{S}^3$ transverse to the NS5-brane. In the geometry (4.12), we have two directions labeled by the vector $\vec{z}$, for which $f_{\vec{z}}(u) = \ell^2$. In this case, there are clearly oscillatory null geodesics. This situation is no different from that in the BMN plane wave. Here we will have identifications between the TIPs and the TIFs for arbitrarily large values of the coordinate u and in particular, we can conclude that the causal boundary in this case is one-dimensional. This is perhaps a little surprising, as one might have expected that the geometry is a direct product of a four dimensional plane wave of the BMN kind (in the sense that two directions have positive mass terms on the world-sheet in the light-cone gauge), with a six dimensional flat space. However, with the persistence of the identifications between the ideal points for the TIPs and the TIFs for arbitratily large values of u, the entire spacetime is visible from $u = \infty$, and hence the causal boundary collapses to a one-dimensional null line.

NL2. We can consider the Penrose limit of the near-horizon geometry of NS5-branes, by looking at null geodesics which in addition to angular momentum on the transverse $\mathbf{S}^3$ also have a radial component. The plane wave geometry resulting from the Einstein frame metric of the NS5-brane is [23],

$$ds_E^2 = -2\,du\,dv - \frac{1}{4u^2}\left(\vec{z}^2 + x^2 + \sum_{i=1}^{5} y_i^2 + b\,u\,\vec{z}^2\right)du^2 + dx^2 + d\vec{z}^2 + \sum_{i=1}^{5} dy_i^2\,, \tag{4.13}$$

with $b = 8\frac{\ell^2}{\sqrt{1-\ell^2}} > 0$. Clearly, the geometry has a null singularity at $u = 0$. For the six directions labelled by (x, y_i), the matrix $f_{ij}(u)$ is diagonal with the entries being $1/4u^2$. As discussed in the example 6 above, in this case the null geodesics along none of these directions exhibit oscillatory behaviour. Along $\vec{z}$ however, it is easy to see that there are oscillatory solutions. Writing $\vec{z} = (z^1, z^2)$ we see that the general solution for the geodesic equation is $z^i(u) = \alpha_1^i\,\sqrt{u}\,J_0(2\sqrt{b}\,u) + \alpha_2^i\,\sqrt{u}\,Y_0(2\sqrt{b}\,u)$, where α_1^i and α_2^i are arbitrary constants. These geodesics exhibit oscillatory behaviour for arbitrarily large values of the u coordinate.

An interesting feature that can be explicitly discussed with example NL2, pertains to the distinction between the string and Einstein frame metrics with regard to the causal structure. Since the string frame metric is related to the Einstein frame metric by a conformal transformation, one expects that for dilaton profiles which are non-singular, the discussion with respect to the causal structure to be identical. In the case of singular behaviour of the dilaton we would encounter additional singularities in the Einstein frame metric. The point that we wish to clarify is that while taking the Penrose limit, one can work either with the string frame or the Einstein frame metric, and the resulting plane wave spacetimes are again related to each other by a conformal rescaling. So unless the Penrose limit induces additional singularities in the dilaton profile, the causal structure for both is identical. In the case of the NS5-brane geometry, if we had taken the Penrose limit for the string frame metric, we would have obtained the plane wave metric and a dilaton,

$$\begin{aligned} ds^2_{str} &= -2\,du\,dv - \ell^2\,\vec{z}^2\,du^2 + dx^2 + d\vec{z}^2 + \sum_{i=1}^{5} dy_i^2 \\ \phi(u) &= -\sqrt{1-\ell^2}\,u\,. \end{aligned} \tag{4.14}$$

The string frame metric is identical to the Nappi-Witten model (4.12), and one concludes that the geodesics in the $\vec{z}$ directions lead to identifications. The dilaton being linear in u leads to a singularity in the spacetime at $u = 0$. This is precisely what we conclude from the Einstein frame metric (4.13).

NL3. The last example we wish to consider is the Penrose limit of the geometry which is holographically related to the non-commutative Yang-Mills theory [27, 28]. In this case the choice of a null geodesic with angular and radial components leads to the plane wave metric [23]

$$ds^2 = -2\,du\,dv - \ell^2\,\left(x^2 + \vec{z}^2 + y_1^2 + g(u)\,y_2^2\right)\,du^2 + dx^2 + d\vec{z}^2 + dy_1^2 + dy_2^2\,, \tag{4.15}$$

where $\ell \leq 1$ and

$$g(u) \equiv \frac{\ell^8 - a^8\sin^8(\ell u) - 2a^4\sin^2(\ell u)\,\cos^2(\ell u)\,\left(a^4\sin^4(\ell u) - 5\ell^4\right)}{\left(\ell^4 + a^4\sin^4(\ell u)\right)^2}\,. \tag{4.16}$$

Here, the spacetime as a whole has identifications between the TIPs and the TIFs, because one has geodesics behaving like harmonic oscillators in all of the transverse directions but y_2, for $\ell \neq 0$. However, it is possible for the identifications to disappear if we consider special limits of the parameters appearing in the metric (4.15), (4.16). This rests on the behaviour of null geodesics along y_2, which are determined by the geodesic equation $\ddot{y}_2(u) + \ell^2\,g(u)\,y_2(u) = 0$. While it is hard to explicitly analyze the geodesic equation analytically, one can numerically integrate it to show that generically there are oscillatory solutions. The situation changes when we consider the strong non-commutativity limit $a \to \infty$ and $\ell = 1$. In this case $g(u) \to -\left(1 + 2\cot^2 u\right)$. This implies that the spacetime is singular at $u = 0, \pi$ and so we take the coordinate $u \in (0, \pi)$. The geodesics along the other transverse directions $(x, \vec{z}, y_1)$ are oscillatory with period 2π (since we take $\ell = 1$). Thus, the identifications between TIPs and TIFs disappear, since they would require a larger separation of the u coordinate than the allowed range $u \in (0, \pi)$.

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As we have seen above and as was pointed out already in [16], there exist examples of spacetimes wherein there is no identification, such as the plane waves arising from the Penrose limit of D0, D1, and D2 brane near horizon geometries. In these cases, the causal boundary comprises of a null line for finite values of u, and past and future null planes (rather than single ideal points) corresponding to $u \to \pm\infty$. This obstructs the statement that the full causal boundary is one-dimensional. More trivially, the flat Minkowski spacetime has no TIP $\leftrightarrow$ TIF identifications, and higher dimensional causal boundary. Since this may appear somewhat puzzling, let us make a few remarks about the distinctions between these plane waves, before proceeding to discuss the more general pp-waves. For simplicity, let us illustrate the point by discussing the Minkowski spacetime example.

4.4 Why is flat spacetime different?

Above, we have seen that if the geodesics have an oscillatory behaviour in v (as a function of u), we can find appropriate sequences which have an accumulation point at large negative values of v. This means that for some u_0 and u_1, all points with $u \geq u_1$ lie in the future of all points with $u \leq u_0$, so that the ideal points corresponding to the TIP $(u_0, v \to +\infty, x^i)$ and the TIF $(u_1, v \to -\infty, x^i)$ get identified.

This identification has a dual role. Firstly, it tells us that instead of two lines of ideal points, one corresponding to TIPs and the other to TIFs of causal curves asymptoting to a finite value of u, we have only a single line, which "winds around" as in figure 2. More importantly, it also tells us that the TIPs of curves asymptoting to an infinite u all give a single ideal point i^+, and similarly, the TIFs of all curves starting from negative infinite u all give rise to a single ideal point i^-. This completes the statement that the causal boundary of such a spacetime is one-dimensional.

Let us now contrast this situation with that of the flat d-dimensional spacetime, which is a special class of plane waves. Since the claim of section 4.1 that

$$\mathrm{TIP}\left[\gamma(u \to u_1)\right] = \{(u, v, x^i) : u \leq u_1\} \tag{4.17}$$

holds for all plane waves (and therefore the flat spacetime), we might deduce that the boundary is parameterized by a single parameter u and thus is one-dimensional. However, this clearly contradicts the obvious fact that a d-dimensional Minkowski spacetime has a $(d-1)$-dimensional boundary. What went wrong? The flaw in the above reasoning stems from the lack of identifications. Since the geodesics in flat space are just straight lines, they can't reach arbitrarily large negative v. In fact, in flat spacetime, v must increase along all future-directed causal curves. The set of ideal points $\{\mathrm{TIP}\left[\gamma(u \to \infty)\right]\}$ corresponding to infinite u is not zero-dimensional as above, but rather a full $(d-1)$-dimensional surface. Put more explicitly, most curves which reach infinite u, unless they simultaneously reach the future timelike infinity i^+, do *not* have the whole spacetime in their past. This is illustrated in figure 3.

Figure 3 shows the Penrose diagram for the flat Minkowski spacetime, with (u, v, r) directions shown explicitly. The whole spacetime is bounded by 2 null cones (the upper corresponding to $\mathcal{I}^+$ and the lower to $\mathcal{I}^-$; in addition there is spatial infinity i^0 and the

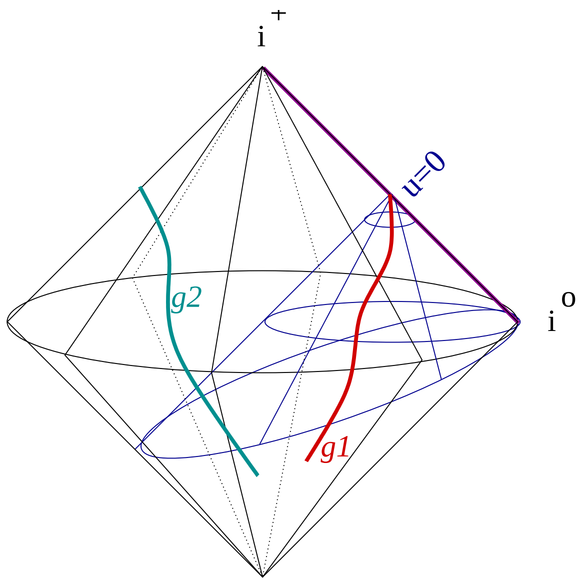

Figure 3: Penrose diagram for Minkowski spacetime. The TIP of a causal curve $g1$ which asymptotes to the $u = 0$ plane lies in the null cone $u = 0$ shown. Since the part of $\mathcal{I}^+$ corresponding to finite u is the upper left line indicated, the curve $g2$ does not asymptote to any finite u.

future/past timelike infinities $i^\pm$ lying at the tips of the cones). The constant u or v null planes are here conformally mapped to null cones inside the spacetime. For example, the $u = 0$ hypersurface is the null cone shown in figure 3. Therefore, the part of $\mathcal{I}^+$ corresponding to *finite* u is the null *line* in the upper right edge of the diagram; the rest of $\mathcal{I}^+$ has infinite u. This is what accounts for the higher dimensionality of the boundary. The claim TIP $[\gamma(u \to 0)] = \{(u, v, x^i) : u \leq 0\}$ is now obvious from figure 3 (where γ is exemplified by the curve $g1$). However, we can also easily see that there are now curves, such as $g2$, which reach infinite u without reaching i^+, and therefore do not contain all of the spacetime in their past.

This property of flat space is also what makes it the only exception to the observation [2] that plane waves are not globally-hyperbolic. In particular, for all non-trivial plane waves, the light cones eventually re-converge in caustics, so that there exists a sequence of null geodesics converging to two distinct, parallel, null geodesics. This property precludes the existence of global Cauchy surfaces. Taking the flat space limit effectively pushes these two null geodesics infinitely far apart; or in other words, the null geodesics do not caustic.

5. Comments on causal structure of pp-waves

We now turn to discussing the causal structure of general pp-waves, with metric as given in (2.2). In order to simplify the discussion we will write this metric in spherically symmetric coordinates,

$$ds^2 = -2\,du\,dv - F(u, r, \Omega)\,du^2 + dr^2 + r^2\,d\Omega^2 \tag{5.1}$$

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where, for simplicity of notation, we will refrain from writing out all the angles and instead content ourselves to the minimal symbol of Ω. The main advantage of this coordinate system is that along any causal curve $\gamma(u \to u_1)$ asymptoting to a finite value of u, only two coordinates, r and v, can diverge. Furthermore, we can show that v must diverge faster than r, so that along any such causal curve v necessarily diverges.

Let us first consider the class of pp-waves which are solutions to vacuum Einstein's equations. Since the Einstein tensor is given by $G_{uu} = \frac{1}{2}\nabla_T^2 F$, where ∇_T^2 is the transverse laplacian, $F(u,r,\Omega)$ of (5.1) must satisfy the transverse Laplace equation, $\nabla_T^2 F = 0$. This is a very remarkable result, since it implies that, due to the linearity of Laplace equation, we may superpose the solutions. In particular, we can decompose F in terms of the $(d-3)$-dimensional spherical harmonics $Y_L(\Omega)$, where $L \equiv \{\ell, m, \ldots\}$:

$$F(u,r,\Omega) = \sum_L \left\{ f_L^+(u)\, r^\ell\, Y_L(\Omega) + f_L^-(u)\, r^{-(d-4+\ell)}\, Y_L(\Omega) \right\}. \tag{5.2}$$

In the neighbourhood of $u = u_0$ and $\Omega = \Omega_0$, where the functional behaviour of $F(u,r,\Omega)$ with respect to the u coordinate is regular[8], we can write

$$F(u_0, r, \Omega_0) \equiv f(r) = \sum_\ell \left\{ f_\ell^+\, r^\ell + f_\ell^-\, r^{-(d-4+\ell)} \right\}. \tag{5.3}$$

As discussed below, there can be singularities at $r = 0$ and/or $r = \infty$.

For pp-wave spacetimes which are not solutions to vacuum Einstein's equations, the discussion in the following subsection will still carry through. At present, it is not completely clear as to what are all the pp-wave solutions, say to just supergravity equations of motion. We will therefore concentrate on some examples that are of interest to string theorists. The main example we consider is the solution written down in [13], a supersymmetric background of IIB string theory, which leads to the $\mathcal{N} = 2$ sine-Gordon theory on the world-sheet in light-cone quantization. The metric for the solution is given by

$$ds^2 = -2\, du\, dv - (\cosh x - \cos y)\, du^2 + dx^2 + dy^2 + dz^i\, dz^i \tag{5.4}$$

where we have reverted back to the cartesian coordinates.

In the following we will first discuss the construction of TIP(F)s for pp-waves (5.1). We show that as in the plane wave case, the TIP of all causal curves asymptoting to $u = u_1$ plane contains all points with $u \le u_1$. Then we turn to the question of identifications between TIPs and TIFs. This will depend on the specific form of the metric, i.e., the choice of the function $F(u,r,\Omega)$. We will show that for the spacetime in (5.4) the TIPs and TIFs get identified as in the BMN plane wave case, and therefore the causal boundary is one-dimensional. On the other hand, for the vacuum pp-waves (5.2) (which are not plane waves) we will see that there are no identifications between the TIPs and TIFs. These spacetimes are also generically geodesically incomplete in contrast to the spacetime (5.4).

[8] Regularity of $F(u,r,\Omega)$ with respect to the angular variables is obvious; and if it so happens that the function $F(u,r,\Omega)$ in (5.1) has a singularity at finite u, say at $u = u_\infty$ then we must impose the additional restriction that $u_0 < u_\infty$.

5.1 TIPs for general pp-waves

We will now proceed to analyze the TIP structure of pp-waves which are of the form given in (5.1). Our main claim will be analogous to (4.2),

$$\mathrm{TIP}\,[\gamma(u \to u_1)] = \{(u, v, r, \Omega) : u \leq u_1\} \tag{5.5}$$

or in words, the TIP of any causal curve γ which asymptotes in the future to $u = u_1$ is given by the set of all points with $u \leq u_1$. As in the plane wave case, if the function $F(u, r, \Omega)$ is singular at some finite $u = u_\infty$, we will require that $u_1 < u_\infty$.

To establish our claim (5.5), we will proceed in parallel with the situation in the case of plane waves. Again it will suffice for us to show that the TIP of any causal curve γ asymptoting to $u = u_1$ contains all points on the surface $u = u_1 - \delta$ for arbitrarily small δ, i.e.,

$$\forall\ \delta > 0\,, \qquad \mathrm{TIP}\,[\gamma(u \to u_1)] \supset \{(u, v, r, \Omega) : u = u_1 - \delta\}\,. \tag{5.6}$$

Once (5.6) is shown, the rest is again established by noting that all points with $u < u_1 - \delta$ are in the past of some point in the $u = u_1 - \delta$ surface, whereas no point with $u > u_1$ can lie in the TIP $[\gamma(u \to u_1)]$,[9] so that by closure, taking $\delta \to 0$ leads to the desired statement that TIP $[\gamma(u \to u_1)] = \{(u, v, r, \Omega) : u \leq u_1\}$.

To prove the claim (5.6), we note that it is equivalent to the claim that for arbitrarily small positive δ, the future of any point $p_0 = (u_0, v_0, r_0, \Omega_0)$ with $u_0 \equiv u_1 - \delta$ necessarily contains a part of any causal curve which asymptotes to $u = u_1$. In other words,

$$\forall\ \delta > 0\,, \quad \exists \quad \varepsilon > 0 \quad \text{s.t.} \quad \gamma(u > u_1 - \varepsilon) \subset I^+(p_0)\,. \tag{5.7}$$

Thus, to prove (5.7), we need to find two separate relations: one delineating an appropriate region $R_\delta \subset I^+(p_0)$ (which we will show must be entered by all causal curves $\gamma(u \to u_1)$), and the other specifying the coordinate relations satisfied by *any* causal curve γ which asymptotes to the $u = u_1$ plane. These relations can be written in terms of coordinate inequalities, essentially relating r and v. Typically, they will take the form

$$v \geq \alpha\, g(r) \tag{5.8}$$

where α is an arbitrary constant and $g(r)$ is a particular function of r, depending on the precise form of the metric. For pp-waves with F growing sufficiently slowly with r, g takes the form $g(r) = r^2$; generically $g(r) \sim F(r)$. If the inequality specifying $\gamma(u \to u_1)$ is more stringent than, i.e. implies, the inequality specifying R_δ (or if they are identical), then obviously any $\gamma(u \to u_1)$ must enter $R_\delta \subset I^+(p_0)$. This is pictorially sketched in figure 4,

[9] Proving that u cannot decrease along a future-directed causal curve here is slightly less trivial than the corresponding proof for plane waves, since there we used the extra planar symmetry. However, we can use a completely different argument, which is similar to that which we used in [18] to argue for absence of horizons in pp-waves: In any finite, but arbitrarily large, region between the origin and infinity, we can bound the function F in (5.1) from above, $F(u, r, \Omega) \leq F_0$. Then any curve which is causal in (5.1) must be causal in a spacetime of the form (5.1) with $F(u, r, \Omega)$ replaced by F_0. But the latter is just the flat spacetime, wherein we know that $\dot{u} \geq 0$, since u is not affected by the coordinate transformation to explicitly flat spacetime. This shows that the same must hold in the present case. An alternate proof is given in e.g. [29].

which denotes the $u-v$ plane. The dashed lines represent the $u=u_0$ and $u=u_1$ planes; p_0 is any point on the former, while γ represents any causal curve asymptoting to the latter. We want to show that any such curve must eventually enter the region $R_\delta \subset I^+(p_0)$.

Let us first consider the relation on the coordinates far along any causal curve γ which asymptotes to the $u=u_1$ plane. As explained above, since γ is future-directed and causal, u must increase along it, so we can parameterize γ by u. Also, since we are interested in the part of the curve above the $u=u_0=u_1-\delta$ plane, and δ will eventually be taken arbitrarily small, we can neglect the u-dependence in the metric, i.e., we can take $F(u,r,\Omega)=F(u_0,r,\Omega)$ in (5.1). From (5.1), the causal relation can then be written as

$$2\,\dot{v} \geq \dot{r}^2 - F(u_0,r,\Omega) + r^2\dot{\Omega}^2 \tag{5.9}$$

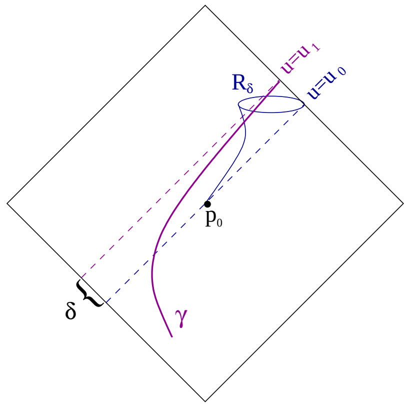

Figure 4: Idea of proof of (5.6); specifically, any causal curve $\gamma(u \to u_1)$ which asymptotes to the $u=u_1$ plane must enter into the region $R_\delta \subset I^+(p_0)$ in the causal future of any point p_0 with its u coordinate $u_0 = u_1 - \delta$, for arbitrarily small δ. The $u-v$ plane is shown on the sketch, but note that this is not a Penrose diagram of a general pp-wave.

where $\dot{} \equiv d/du$. Now, since γ asymptotes to a finite-u plane, some other coordinate must diverge. As for plane waves, it can be easily shown that $v(u) \to \infty$ as $u \to u_1$. This implies that $\dot{v} \to \infty$ as $u \to u_1$ as well. On the other hand, $r(u)$ can behave in several distinct ways; so we will now analyze these cases in turn.

Case 1. $r(u)$ remains finite as $u \to u_1$. Since v diverges, this means that we can satisfy the inequality (5.8) with $g(r)=r^p$ for any α and p, far enough along γ.

Alternately, in **Case 2**, $r(u) \to \infty$ as $u \to u_1$. Now it will be relevant how the function $F(u_0,r,\Omega)$ behaves as r gets large. There are, in turn, several distinct possibilities.

Case 2a. $\dot{r}^2 - F + r^2\,\dot{\Omega}^2 \longrightarrow \dot{r}^2 + r^2\,\dot{\Omega}^2$ as $u \to u_1$, i.e., as $r \to \infty$. Then the causal relation (5.9) may be reproduced by the flat fiducial metric $ds^2 = -2\,du\,dv + (1-\bar{\varepsilon})\left(dr^2 + r^2\,d\Omega^2\right)$. This can be integrated to yield the finite difference relation along γ,

$$2\,\Delta v\,\Delta u \geq \left[(\Delta r)^2 + r^2\,(\Delta\Omega)^2\right]\,(1-\tilde{\varepsilon}) \tag{5.10}$$

where $\tilde{\varepsilon}$ encodes the error in neglecting F, which can be made arbitrarily small by going far enough along γ. Now, for large v, $\Delta v \approx v$, which again becomes arbitrarily accurate by going far enough along γ. Also, since this inequality is most stringent if the final r, and therefore Δr, is large, we can also take $\Delta r \approx r$. Thus (5.10) yields the inequality

$$v \geq \frac{(1-\tilde{\varepsilon})}{2\,\Delta u}\left[1+(\Delta\Omega)^2\right]\,r^2\,. \tag{5.11}$$

Since $(\Delta\Omega)^2$ is a finite $\mathcal{O}(1)$ number, we can rewrite this as

$$v \geq \alpha\,r^2 \tag{5.12}$$

where α can be arbitrarily large by choosing Δu sufficiently small. In the asymptotic region along γ, this is directly related to choosing ε sufficiently small in (5.7).

Case 2b. $\dot{r}^2 - F + r^2\,\dot{\Omega}^2 \longrightarrow -F > 0$ as $u \to u_1$. The inequality on F should be taken to mean the inequality for the asymptotic region along γ, i.e., $F(u_0, r \to \infty, \Omega_\gamma)$, which has a definite sign. (The sign of F may change along different curves γ which reach infinite r in different directions, i.e. for different Ω.) Then (5.9) implies that $2\,\dot{v} \geq -F \gg \dot{r}^2 + r^2\,\dot{\Omega}^2$, and the same bound (5.12) as derived above must apply. However, this is not sufficient; we can in fact derive a much stronger bound. By considering the corresponding finite difference relation, $2\frac{\Delta v}{\Delta u} \geq |F|$, we arrive at the relation

$$v \geq \varepsilon' \,|F(u_0, r, \Omega)| \tag{5.13}$$

where ε' is small (corresponding to small Δu), but fixed.

Case 2c. $\dot{r}^2 - F + r^2\,\dot{\Omega}^2 \longrightarrow -F < 0$ as $u \to u_1$. When $F > 0$, we encounter the most trivial case. This is because in the relevant regions, causality in our spacetime (5.1) is more stringent than in the flat spacetime. So by the same type of argument as in [18], we can immediately see that the claim TIP $[\gamma(u \to u_1)] = \{(u, v, r, \Omega) : u \leq u_1\}$ must be satisfied.

Note that when $\frac{(\dot{r}^2 + r^2\dot{\Omega}^2)}{|F|} \to$ const as $r \to \infty$, as in the plane wave case, we can use the same construction as in Case 2a or Case 2b, with appropriate modifications of the constants involved.

To summarize, we have obtained two relations between the coordinates far enough along any causal curve which asymptotes to the $u = u_1$ plane, given by (5.12) and (5.13) in the respective cases depending on the form of F in the pp-wave spacetime (5.1).

We now turn to the second part of our proof, namely constructing the region R_δ in the causal future of a point p_0 lying on the $u = u_0$ plane. Since we want to ascertain that all the curves considered above enter this region, it suffices to consider the part of the causal future of p_0 with u between u_0 and u_1, and the coordinate v large. To that end, we may again make the approximation $u = u_0$ in the metric (5.1).

We want to show that if at some point $p_f = (u_f, v_f, r_f, \Omega_f)$, the inequality (5.12) or (5.13) is satisfied (for u_f sufficiently close to u_1 and v_f sufficiently large), then $p_f \in I^+(p_0)$. To this end, it suffices to construct a causal curve $\mathcal{C}$ from p_0 to p_f. We will construct such a curve in three stages, $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$, by introducing two convenient intermediate points, q_0 and q_f. In particular, q_0 will have the final angle, Ω_f, but $r = r_0$, while q_f will be chosen so as to saturate the inequality (5.12) or (5.13), and all coordinates except for v will match those of p_f. Pictorially, we construct causal curves between respective points as follows:

$$\begin{aligned}
&p_0 = (u_0,\ v_0,\ r_0,\ \Omega_0)\\
&\downarrow\ \mathcal{C}_1\\
&q_0 = (u_q,\ v_q,\ r_0,\ \Omega_f) \qquad \text{with } u_0 < u_q < u_f, \quad \text{and} \quad v_0 < v_q < v_f\\
&\downarrow\ \mathcal{C}_2\\
&q_f = (u_f,\ v = \alpha\, r_f^2,\ r_f,\ \Omega_f) \qquad \text{for case 2a}\,,
\end{aligned}$$

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$$(\text{or } q_f = (u_f,\ v = \varepsilon' \, |F(u_0, r, \Omega_f)|,\ r_f,\ \Omega_f) \text{ for case 2b})$$
$$\downarrow\ \mathcal{C}_3$$
$$p_f = (u_f,\ v_f,\ r_f,\ \Omega_f) \tag{5.14}$$

where the constants appearing in q_f are to be picked later. It is easiest to construct the last curve, $\mathcal{C}_3$, from q_f to p_f, because v is the only coordinate along $\mathcal{C}_3$ which needs to change. Since $\left(\frac{\partial}{\partial v}\right)^a$ is a null Killing field, it is readily apparent that q_f and p_f are in fact connected by a null geodesic following the orbit of $\left(\frac{\partial}{\partial v}\right)^a$.

Let us now proceed with constructing $\mathcal{C}_1$, the curve from p_0 to q_0, which takes us around the origin. To this end, we can choose the coordinates along $\mathcal{C}_1$ to satisfy

$$\begin{aligned} \dot{r} &= 0 \quad \Rightarrow \quad F = F(u_0, r_0, \Omega) \equiv F_0(\Omega) \\ 2\,\dot{v} &= A \\ r_0^2\,\dot{\Omega}^2 &= A + F_0(\Omega) \end{aligned} \tag{5.15}$$

where A is some positive constant, chosen such that a solution exists. In particular, since the l.h.s. of the last equation must be positive, whereas F_0 can be negative, we require that $A > -F_0(\Omega)$. This is possible, since $F_0(\Omega)$ remains bounded at any constant r. Clearly, such a curve $\mathcal{C}_1$ satisfying (5.15) exists, and from (5.9), it is null and therefore causal. The remaining requirement on $\mathcal{C}_1$ is that u_q lie between u_0 and u_f, i.e., that $\mathcal{C}_1$ rounds the origin sufficiently fast; but this is easily satisfied. In fact, we can make u_f arbitrarily close to u_0 by taking A large enough.

Finally, we need to construct the intermediate curve $\mathcal{C}_2$ between q_0 and q_f, which takes us to large r while saturating the requisite inequality on v. This can be achieved by the following conditions:

$$\begin{aligned} \dot{\Omega} &= 0 \quad \Rightarrow \quad \Omega = \Omega_f \\ B\,\dot{r}^2 &= F(u_0, r, \Omega_f) \equiv f(r) \\ 2\,\dot{v} &= (1-B)\,\dot{r}^2 = \frac{(1-B)}{B}\,f(r) \end{aligned} \tag{5.16}$$

where B is some constant, $B < 1$. Note that, unless $f = 0$, by taking B sufficiently small, our curve $\mathcal{C}_2$ can reach[10] sufficiently large values of r and correspondingly v. Now we can repeat the same finite-difference argument as above: for instance, in case 2a, for sufficiently small Δu, the last equation in (5.16) implies that $2\,\Delta v\,\Delta u = (1-B)\,(\Delta r)^2\,(1-\tilde{\varepsilon})$, so that choosing $\alpha = \frac{(1-B)\,(1-\tilde{\varepsilon})}{2\,\Delta u}$, we obtain the necessary relation (5.12) for v in terms of r.

We can see the above construction more explicitly by considering several illustrative examples. Let us first confirm the validity of $\mathcal{C}_2$ for a simple plane wave example,

[10] Construction of such a curve could be problematic in regions where $f(r)$ changes sign. To specify the curve completely, we therefore pick the sign of B depending on the sign of $f(r)$ near $r = r_0$, such that $\dot{r}^2 \geq 0$. Now suppose $f(r)$ changes sign for some $r = r_i > r_0$. At r_i we also flip the sign of B and continue with the construction of the causal curve. In other words, we can always solve the equation $\dot{r}^2 = |f(r)/B|$, for which the existence of a solution is guaranteed. Of course, such a curve may have higher derivative discontinuities; but we can then smooth these out.

$F(u_0, r, \Omega_f) = r^2$. Then (5.16) implies that $r(u) = e^{u/\sqrt{B}}$ and $v = \frac{(1-B)}{4\sqrt{B}}\, e^{2u/\sqrt{B}} = \alpha\, r^2$ for $\alpha \equiv (1-B)/4\sqrt{B}$, which is clearly consistent with (5.12). As a second example, let us consider the more "dangerous" type of behavior, such as $F(u_0, r, \Omega_f) = -r^4$. To simplify notation, let us pick the arbitrary constants $r_0 = 1$ and $u_0 = 0$. Then $r(u) = \sqrt{|B|}/(\sqrt{|B|} - u)$, so that from integrating the last equation of (5.16) and re-expressing $v(u)$ in terms of r, $v = \frac{|B|\,(1-B)}{6}\, r^3$. Clearly, for sufficiently small $|B|$, we can satisfy $\frac{|B|\,(1-B)}{6} \le \varepsilon'\, r_f$, which yields a relation consistent with the necessary inequality (5.13). As a final example (also in the 2b category), when $F \sim -e^r$, (5.16) gives $v = \beta(B)\, e^{r/2} \ll \varepsilon'\, e^r$ for $\beta(B) \ll \varepsilon'\, e^{r_f/2}$. Again, this implies that any causal curve which asymptotes to the $u = u_1$ plane, that necessarily satisfies $v \ge \varepsilon'\, e^r$, enters into the future of p_0, given by $v \ge \beta(B)\, e^{r/2}$.

Thus, we have seen that by putting together[11] three null curves, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, we can construct a causal curve $\mathcal{C}$ which connects p_0 with p_f. The points $p_f \in R_\delta$ by construction all satisfy the requisite relation insuring that all causal curves which asymptote to $u = u_1$ must enter R_δ. This completes the proof of the claim (5.5), that TIP $[\gamma(u \to u_1)] = \{(u, v, r, \Omega) : u \le u_1\}$. **(QED)**

5.2 Identifications for general pp-waves

The preceding subsection established the claim (5.5) that the TIP of any causal curve which asymptotes to finite u_1 is given by the set of all points with $u \le u_1$. By time-reversing our statements, the analogous result holds for the TIFs. Thus, this part of $\mathcal{I}$ is parameterized by a single parameter u, and is therefore one-dimensional. However, as cautioned in section 4.4, this does not automatically imply that the full causal boundary of pp-waves is one-dimensional, as demonstrated by the Minkowski spacetime example. Note however, that the construction of TIPs for general pp-waves at finite u enables us, as in Sec 4.2, to infer the absence of horizons in these spacetimes, reconfirming the claims of [18]. What remains to be examined, in order to determine the full causal structure, are the identifications between the TIPs and TIFs.

In order to ascertain which ideal points need to be identified, based on causal properties alone, we need to isolate the ideal points which have the identical causal past and future. Let us start by considering ideal points which correspond to TIPs. Here, since the ideal points are labelled by a single coordinate u, it is clear that there are no identifications between the ideal points. Considering two values of u, say u_1 and u_2, with $u_1 < u_2$ it is clear that the causal past of all causal curves asymptoting to $u = u_1$ is distinct from that of causal past of curves asymptoting to $u = u_2$. By time-reversal, similarly there are no identifications between the ideal points corresponding to TIFs.

The main issue then is whether there exist any identifications between the ideal points associated with TIPs and TIFs. To this end, we can employ the same technique as for the plane wave case. To wit, if there exists a sequence of causal curves which emanate from *any* point in the $u = u_0$ plane, say $p_0 = (u_0, v_0, x_0^i)$, and have an accumulation

[11] To obtain a smooth curve, we would need to smooth out the sharp edges. Typically, this can be done in such a way that the new curve is timelike, since we can always perturb the curves to introduce a small timelike component.

curve which reaches the point $p = (u_1, v = -\infty, x^i)$, where x^i are arbitrary but finite, and $u_1 > u_0$ is the smallest value of u for which such an accumulation curve exists, then we would identify TIP $[\gamma(u \to u_0)]$ with the TIF $[\gamma(u \to u_1)]$. If such identifications persist for arbitrarily large values of u, then we see that the structure at $u = \infty$ is different from e.g. that of Minkowski space, and we will have a one-dimensional causal boundary. If we encounter singular behaviour of the functions $F(u, x^i)$ at finite values of $u = u_\infty$, then the ideal points associated with the $u = u_\infty$ plane would correspond to a null singularity. Also there are situations with regular behaviour of $F(u, x^i)$ with respect to u, but without any identifications.

To complete the program of the causal structure for pp-waves, we therefore need to understand in which cases are there appropriate causal curves which allow for identifications between the ideal point associated with a TIP and a TIF. To this end, it is useful to analyse in detail the behaviour of the geodesics in the pp-wave spacetimes. If we can establish that the behaviour of the geodesics is oscillatory for arbitrarily large values of u, then, as in the plane wave case, with a clever choice of initial conditions we can establish that causal communication to arbitrarily large negative values of v is possible. Note that in the absence of caustics, null geodesics bound the future light cone emanating from a point. Hence, if we can show that it is not possible for null geodesics to reach large negative v, then causal curves will also be unable to reach $v \to -\infty$.

The derivation of the geodesic equations is presented in the appendix A; we have

$$\ddot{x}^i + \frac{1}{2}\partial_i F(u, x^i) = 0 \tag{5.17}$$

$$2\dot{v} = -F(u, x) + \sum_i (\dot{x^i})^2 \tag{5.18}$$

Note that unless $F(u, x)$ takes the plane wave form $f_{ij}(u)\, x^i x^j$, $\dot{v}$ is not a total derivative, so we can't write a general formula for v analogous to (4.10), without first solving the x-equations (5.17). In what follows, we will analyze these geodesic equations for certain special cases and show that, while there are identifications in certain examples, such as the pp-wave background discussed in [13] with metric as given in (5.4), generic vacuum pp-waves do not admit any identifications and are singular (geodesically incomplete). We will first consider examples where we have identifications between the ideal points, as they are most analogous to the hitherto analyzed plane wave examples.

PP1. Our first example is the pp-wave background (5.4), where $F(u, x^i) \equiv \cosh x - \cos y$. This is interesting from the string theory point of view since it leads to an integrable sine-Gordon like theory on the world-sheet in light-cone gauge. The geodesic equations in this case need to be analysed only for the coordinates, x and y, $F(u, z^i)$ is independent of the remaining coordinates. The geodesic equations (5.17) read,

$$\begin{aligned} \ddot{x} + \frac{1}{2}\sinh x = 0 \quad &\Rightarrow \quad \dot{x}^2 + \cosh x = \alpha \\ \ddot{y} + \frac{1}{2}\sin y = 0 \quad &\Rightarrow \quad \dot{y}^2 - \cos y = \beta\,, \end{aligned} \tag{5.19}$$

with arbitrary constants of integration α and β. Given the solution to (5.19), we can determine $v(u)$ from the equation

$$2\,\dot{v} = -F(x,y) + \dot{x}^2 + \dot{y}^2 = \alpha + \beta - 2\cosh x(u) + 2\cos y(u)\,. \tag{5.20}$$

Clearly, the geodesics in the y direction are not going to be oscillatory for $\beta > 1$, while those in the x direction are going to oscillate. The geodesic motion in the x direction is just the motion of a particle in a $\cosh x$ potential, with a fixed total energy α. Given a fixed energy, the particle can rise up to the potential upto the level specified by the energy and then starts to roll back down. Since the potential is reflection symmetric, this process continues indefinitely.

From any point $p_0 = (u_0, v_0, x_0, y_0, z_0^i)$ in the spacetime (5.4), we can find a sequence of null geodesics γ_n, parameterised by α_n, such that along γ_n we have $v_n \to -\infty$. This is made possible by taking α arbitrarily large in the geodesic equations (5.19), (5.20). We illustrate this in figure 5, where we plot v as a function of u, for two different values of α. The higher curve corresponds to the larger value of α. Thus, it is possible in this example to find identifications between the TIPs and the TIFs and in particular, we can claim that *the causal boundary of the pp-wave (5.4) is one-dimensional.*

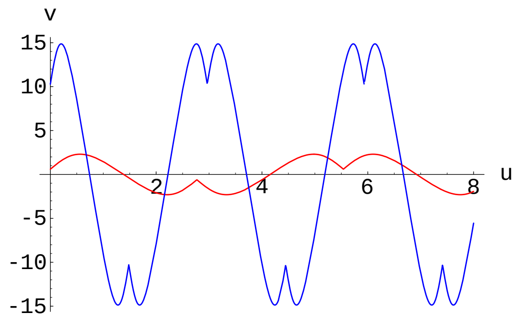

Figure 5: Plot of $v(u)$ for (5.4) spacetime for null geodesics moving in v and x directions only. α in (5.19) is chosen to be $\cosh 3$ and $\cosh 5$. The $\alpha = \cosh 5$ curve attains larger values of $|v(u)|$.

As for the BMN plane wave case, one can also show that this one-dimensional boundary is locally null. This can be achieved by demonstrating that two ideal points P and Q are causal, i.e., there exists a sequence of points $P_n \to P$ with all P_n timelike separated from Q, but there exists a sequence of points $Q_n \to Q$ with all Q_n spacelike separated from P.

Before proceeding to other examples, we wish to point out an interesting feature associated with this example. It is a common belief that plane waves can be the only geodesically complete pp-waves. This relates to an unproven conjecture of Ehlers and Kundt [30] as stated in [31]. The pp-wave (5.4) is, however, *geodesically complete.*[12] From the geodesic equations (5.19) it is clear that the confining nature of the potential in the x direction keeps the geodesic from running off to infinity at finite affine parameter. Along the y direction, one has oscillations with finite amplitude superposed on linear growth with time, which also doesn't asymptote to infinity at finite values of the affine parameter.

The solutions to the geodesic equations (5.19), may be written down explicitly in terms of the Elliptic functions as:

$$z^i(u) = \kappa\,(u + u_0)$$

[12] In fact, we will see later that there do exist vacuum pp-waves which are geodesically complete in $d \geq 5$ dimensions.

$$
\begin{aligned}
u+u_0 &= -\frac{2i}{\sqrt{\alpha-1}}\,\mathrm{F}\left(\frac{i\,x(u)}{2},-\frac{2}{\alpha-1}\right)\\
u+u_0 &= \frac{2}{\sqrt{1+\beta}}\,\mathrm{F}\left(\frac{y(u)}{2},\frac{2}{\beta+1}\right)
\end{aligned}
\tag{5.21}
$$

$$
\begin{aligned}
v(u) &= \left(\kappa^2+\eta\right)(u+u_0)-\\
&\quad -\frac{2i}{\sqrt{\alpha-1}}\left[(\alpha-1)\,\mathrm{E}\left(\frac{i\,x(u)}{2},-\frac{2}{\alpha-1}\right)-\frac{\alpha}{2}\,\mathrm{F}\left(\frac{i\,x(u)}{2},-\frac{2}{\alpha-1}\right)\right]+\\
&\quad +\frac{2}{\sqrt{1+\beta}}\left[(\beta+1)\,\mathrm{E}\left(\frac{y(u)}{2},\frac{2}{\beta+1}\right)-\frac{\beta}{2}\,\mathrm{F}\left(\frac{y(u)}{2},\frac{2}{\beta+1}\right)\right].
\end{aligned}
\tag{5.22}
$$

In the above $\eta=(\pm1,0)$ denote timelike, spacelike, and null geodesics, respectively, and $\mathrm{F}(\phi,m)$ and $\mathrm{E}(\phi,m)$ are the elliptic integrals of the first and second kind, respectively [22].

PP2. Motivated by the example in **PP1**, we can try to characterize the cases with identifications. Let us assume for simplicity that $F(u,x^i)$ is independent of u and can be written as sum of functions, each of which depends on single coordinate i.e., $F(u,x^i)=\sum_j f_j(x^j)$. Essentially we want $\partial_i\partial_j F(u,x^i)=0$ and $\partial_u F(u,x^i)=0$. A prototypical example of such a spacetime is of course (5.4). The idea is to decouple the geodesic equations (5.17) so that they may be analyzed separately. If we can find oscillatory behaviour along any one of the coordinates, then we will be able to conclude that there are identifications between TIPs and TIFs, unless we are hampered by some of the other coordinates leading to singularities.

We concentrate on a single coordinate, $x\in(-\infty,\infty)$, and write the corresponding function as $f(x)$. We also assume that all other coordinates are sufficiently well behaved i.e., the geodesics in these directions are complete. Concentrating on geodesics along x, geodesic equations (5.17) and (5.18) read

$$
\begin{aligned}
\ddot{x}+\frac{1}{2}\frac{\partial f(x)}{\partial x}=0 \quad &\Rightarrow \quad \dot{x}^2+f(x)=\alpha\\
2\,\dot{v}=\dot{x}^2-f(x) \quad &\Rightarrow \quad \dot{v}=\frac{\alpha}{2}-f(x)
\end{aligned}
\tag{5.23}
$$

with $\dot{}\equiv d/du$. We can interpret the geodesic equations as motion of a particle along the x direction in a potential $f(x)$ and total energy α.

From the first equation in (5.23) it is clear that as long as $f(x)$ is bounded from below, the particle doesn't reach $x=\infty$ in finite time u. Therefore such spacetimes are geodesically complete. This, however, does not mean that spacetimes with $f(x)$ not bounded from below are geodesically incomplete. For example, when $f(x)=-x^2$, the particle trajectory as a function of time is exponential, so it takes infinite time to reach out to the asymptotic regions. If $f(x)$ is bounded from below and in addition $f(x\to\pm\infty)\to+\infty$, then there are identifications between the TIPs and TIFs, and the causal boundary is one-dimensional.

To exemplify the situation, consider the case when $f(x)=x^{2n}$ for some integer $n>1$. In this case we have the geodesic equation $\dot{x}^2+x^{2n}=\alpha$, which has oscillatory solutions. In particular, $u=\frac{x(u)}{\sqrt{\alpha}}\,\mathrm{F}\left(\frac{1}{2n},\frac{1}{2};1+\frac{1}{2n};\frac{x(u)^{2n}}{\alpha}\right)$ with $\mathrm{F}(a,b;c;x)$ being the hypergeometric function and $v(u)$ can be expressed implicitly as a function of u. One can verify numerically that the oscillatory behaviour implies identifications between the TIPs and TIFs for arbitrarily large values of u, leading to a one-dimensional causal boundary.

On the other hand, if one were to choose $f(x) = x^{2n+1}$ for some integer $n > 1$, then the spacetime is geodesically incomplete. This is because there is an instability for $x < 0$ and the potential $f(x)$ is unbounded from below. So the fiducial particle moving in this potential is going to run away to infinity in finite time. However, since along these geodesics $v \to +\infty$, there aren't any causal curves or geodesics in these spacetimes which will allow us to causally communicate from any point $p_0 = (u_0, v_0, x_0)$ to $p = (u_1, v = -\infty, x)$ with finite x and $u_1 > u_0$. From this we conclude that the ideal points correspond to singularities and that there are no identifications between them.

PP3. Our next example will be vacuum pp-waves, which we will take to be of the form presented in (5.1) and (5.2). We will show that vacuum pp-waves are generically geodesically incomplete; the exceptions being the flat space and plane waves (2.3) with regular functions $f_{ij}(u)$, and the monopole pp-wave in dimension $d \geq 5$. As we see from (5.2), one can have singular behaviour of g_{uu} in two distinct ways; either the metric component is ill-behaved as $r \to \infty$, or there is a singularity as $r \to 0$. Since we can superpose solutions, in generic vacuum pp-waves both such behaviours are possible. We will investigate two cases, wherein the singularities are either only at $r = 0$ or only at $r = \infty$.

Consider vacuum pp-waves (5.1) with the function $F(u, r, \Omega)$ as in (5.2), with $f_L^+(u) \equiv 0$ and $f_L^-(u) \neq 0$ for some $L = \ell_1 \neq 0$. For simplicity we further assume that $\frac{d}{du} f_{\ell_1}^-(u) = 0$. A particular example of such a spacetime would be

$$ds^2 = -2\, du\, dv - \frac{1}{r^4} \left(5\cos^3\theta - 3\cos\theta\right) du^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \qquad (5.24)$$

In (5.24) there are geodesics which hit the singularity $r = 0$ at a finite affine parameter, causing the spacetime to be geodesically incomplete. Furthermore, since the spacetime rapidly approaches flat space as $r \to \infty$, we find that there is no oscillatory behaviour of null geodesics, implying that there exists no reason from causal properties to identify ideal points corresponding to different TIPs with those corresponding to TIFs. This is the generic behaviour of vacuum pp-waves in arbitrary dimension so long as $f_L^+(u) \equiv 0$, with one notable exception. The monopole solution with $F(u, r, \Omega) = +\frac{1}{r^{d-4}}$ in d-dimensions is geodesically complete. This must be so since $F(r)$ is bounded from below. One can check explicitly that geodesics do not end at a finite affine parameter. On the contrary taking $F(u, r, \Omega) = -\frac{1}{r^{d-4}}$ leads to geodesically incomplete spacetimes. In appendix B we present some solutions to the geodesic equations (5.24).

Now let us turn to the example when $f_L^-(u) \equiv 0$ and $f_L^+(u) \neq 0$. We will in addition also assume $\frac{d}{du} f_L^+(u) = 0$. As an example, one can take the five-dimensional vacuum pp-wave

$$ds^2 = -2\, du\, dv - r^3 \left(5\cos^3\theta - 3\cos\theta\right) du^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \qquad (5.25)$$

If we try to look for geodesics at a constant value of the angular coordinate, i.e., geodesics with only a radial component, we find that they exist for $\theta = 0$ or for $\cos\theta = \pm\sqrt{1/5}$. Consider the radial null geodesic from $(u = 0, v_0, r_0, \theta_0, \phi_0)$, with constant values of the angular variables given by arbitrary ϕ_0 and $\cos\theta_0 = \sqrt{1/5}$. This radial null geodesic reaches infinite values of the radial coordinate in finite affine parameter. On the other hand, a radial null geodesic sitting at constant angular variables $(\theta_0 = 0, \phi_0)$ with arbitrary ϕ_0 exhibits oscillatory behaviour. Such is immediately obvious by just considering the function $F(r,\theta) = r^3\left(5\cos^3\theta - 3\cos\theta\right)$ at constant values of $\theta = \theta_0$. It is clear that for all values of θ_0 such that $\cos^2\theta_0 < 3/5$, the radial potential $F(r,\theta_0)$ is essentially negative definite. First of all, the presence of at least one geodesic which diverges at a finite affine parameter implies that the spacetime is geodesically incomplete. Furthermore, in order for us to claim the existence of identifications between the ideal points, it is not sufficient to find special null geodesics which exhibit oscillatory behaviour. What we need to show is that starting from *any* point on a plane of constant $u = u_0$, we can causally connect to a plane of constant $u = u_1$ with $u_1 > u_0$ with the coordintate $v \to -\infty$. This is not possible for generic geodesics in the spacetime (5.25). Hence, we conclude that there are no identifications.

PP4. Our last example is the four dimensional vacuum pp-wave spacetime[13]

$$ds^2 = -2\,du\,dv - \sin x\, e^y\, du^2 + dx^2 + dy^2\,. \tag{5.26}$$

The geodesic equations (4.9) in this spacetime read

$$\begin{aligned} \ddot{x} + \frac{1}{2}\cos x\, e^y &= 0 \\ \ddot{y} + \frac{1}{2}\sin x\, e^y &= 0\,. \end{aligned} \tag{5.27}$$

Because the function $F(x,y) = \sin x\, e^y$ is not a positive definite function in the variable x, we find that the potential for motion in the y direction is unbounded from below. This causes the geodesics to diverge at finite values of u. In figure 6 we plot the geodesics $(x(u), y(u))$ for some generic initial conditions to illustrate the point.

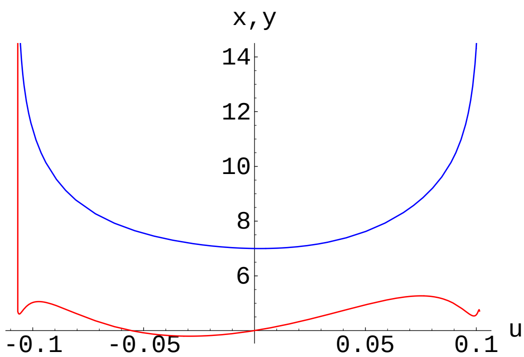

Figure 6: Geodesics $\{x(u), y(u)\}$ for the spacetime (5.26). We present solutions to (5.27), for some generic initial conditions to exhibit geodesic incompleteness.

From examples **PP3** and **PP4**, it is clear that for vacuum pp-waves there are no identifications between TIPs and TIFs. The reason that this happens is that for vacuum solutions we need the function $F(u, x^i)$ in (2.2) to be harmonic. The only harmonic function that is bounded from below is the constant function (or the monopole solution), i.e., with no angular dependence. All other harmonic functions have domains where they either diverge to $+\infty$ or to $-\infty$. The divergence towards positive infinity is acceptable, since this leads to confining potentials; but the divergence towards negative infinity causes runaway behaviour and thence generically to geodesic incompleteness. The situation is somewhat mitigated if $F(u, x^i)$ grows slower

[13] We thank Chris Hillman for this example.

than $(x^i)^3$ in the transverse coordinates (here we are assuming that $F(u,x^i)$ is a regular function of u; if it is not we have null singularities at $u = u_\infty$, where $F(u \to u_\infty, x^i) \to \pm\infty$). For $F(u,x^i) = a(u) + b_i(u)\, x^i$ we have flat space, and for $F(u,x^i) = f_{ij}(u)\, x^i\, x^j$ we have plane waves, which are geodesically complete. Geodesic completeness for generic plane fronted waves with sub-quadratic growth of $F(u,x^i)$ was recently discussed in [29].

6. Discussion

In the present paper we have analyzed in some detail the causal structure of pp-wave spacetimes. This work generalises the recent discussion of causal structure of certain plane wave spacetimes [15, 16]. Since most pp-wave spacetimes are not conformally flat, one cannot use the simple trick of reading off the causal structure from the conformal rescaling into Einstein Static Universe. Instead, as [16], we use the technique developed by Geroch, Kronheimer, and Penrose [17], which identifies the causal structure by looking at future(past) endless causal curves in the spacetime and assigns ideal points to these curves. The ideal points have the same causal future(past) as the curve. Having assigned ideal points to the spacetime manifold, one has to ensure that this assignment is minimal. This requires that different ideal points have distinct causal domains of influence. A more stringent requirement is that upon adjoining the minimal set of ideal points to our spacetime, we obtain a smooth Hausdorff manifold.

As we have seen, for plane waves where the metric component g_{uu} doesn't approach flat space too rapidly, such as polynomials, trigonometric, or hyperbolic functions $f_{ij}(u)$ in (2.3), the causal boundary is one-dimensional and null. On the other hand, spacetimes where the approach to flat space is sufficiently rapid in u, the causal boundary is higher dimensional, as in Minkowski space. The plane wave spacetimes obtained as Penrose limits of supergravity backgrounds dual to non-local theories, such as little string theory and non-commutative gauge theory, generically have a one-dimensional causal boundary which is null. In the case of non-commutative theories, in certain corners of parameter space, the spacetime becomes singular and loses the property of having a one-dimensional boundary.

For pp-wave spacetimes (2.2), the situation is more diverse, since now the function $F(u,x^i)$ can exhibit singular behaviour in the transverse coordinates as well as in u. For the spacetime (5.4) discussed in [13], we find that the causal boundary is one-dimensional and null. More generally, for geodesically complete pp-wave spacetimes wherein the geodesic equations decouple, we have a one-dimensional causal boundary if the following is satisfied: $F(u,x^i) = \sum_j f_j(x^j)$ and at least one of the f_j's diverges to plus infinity as $x^j \to \pm\infty$, in other words, oscillations of $x^j(u)$ induce an oscillatory behaviour in $v(u)$. In the case of generic vacuum pp-waves which are not plane waves or flat space, we find that the spacetimes are geodesically incomplete and the causal boundary is no longer one-dimensional. In addition to explicitly constructing the causal structure for these spacetimes, we have demonstrated that there are no horizons, since any point in these spacetimes is causally connected to infinity, thereby putting the arguments of [18] at a level of greater rigor.

One interesting aspect uncovered in our analysis relates to geodesic completeness of pp-waves. The common understanding is that plane waves are the only geodesically complete subset of pp-waves [4, 31, 30]. We have shown that the monopole vacuum pp-wave and some non vacuum pp-waves, such as (5.4), are geodesically complete. In fact, we can demonstrate geodesic completeness for pp-wave spacetimes wherein the geodesic equations decouple, and the functions appearing in g_{uu} are bounded from below for each of the transverse space coordinates. This criterion is satisfied if, for $F(u, x^i) = \sum_j f_j(x^j)$, the functions $f_j(x^j)$ are either bounded from below or $|f_j(x_j)|$ grow slower than $(x^j)^3$. Interestingly, the geodesic equations (5.17) are identical to the world-sheet equations of motion, for strings propagating in the background (2.2). For a well defined world-sheet sigma model, we would require that the 'potential function' $F(u, x^i)$ be bounded from below. The connection between classical equations of motion of the string world-sheet, and geodesic equations for null geodesics in the spacetime, is quite suggestive of the fact that the string propagation in geodesically incomplete pp-wave backgrounds suffers from some pathological behaviour.

In general relativity, a geodesically incomplete spacetime, i.e., a spacetime wherein at least one in-extendible geodesic ends at a finite value of its affine parameter, is by definition a singular spacetime. Conversely, if all geodesics can be shown to be complete, as in the above example, such a spacetime is not singular, despite some Riemann curvature tensor components diverging. In the cases when there are singularities, the usual expectation is that we cannot extend the spacetime past a singularity, since observers cannot pass through. It is interesting to note that the last statement, however, need not be true. As we have seen in the vacuum pp-wave spacetimes, there can be special geodesics with very non-generic properties. As a simpler example, consider the (non-vacuum) plane wave spacetime (2.3) with $f_{ij}(u) = \frac{1}{u^2}\,\delta_{ij}$, i.e.,

$$ds^2 = -2\,du\,dv - \frac{1}{u^2}\,(x^i)^2\,du^2 + dx^i\,dx^i\,. \tag{6.1}$$

The corresponding geodesic equation is $\ddot{x}^i + \frac{1}{u^2}\,x^i = 0$, so that generic geodesics end at $u = 0$. However, the special geodesic with initial conditions $x^i(u_0) = 0$ and $\dot{x}^i(u_0) = 0$ can be extended through $u = 0$. In fact, we can use the planar symmetry of plane waves to generate a whole family of such geodesics. However, for a physical observer, the tidal forces still diverge at $u = 0$ in this example.

Singular plane wave geometries, such as discussed above, may provide useful toy models for studying singularity resolution in string theory. In particular, there are two useful features of these spacetimes. Firstly, they can be viewed as a Penrose limit of some singular "parent" spacetime, wherein the singularity can be timelike, spacelike, or null. However, in taking the Penrose limit, any null geodesic which terminates on these parent singularities, universally turns them into null singularities. This constitutes the second useful feature: the singularity in plane waves must be null, since it appears at a given u value and extends in the v direction. For a discussion on singularities in plane waves see [4] and more recently [32, 33]. This universal behaviour of singularities in the Penrose limit is tantalizing and might hold some clues to resolving spacelike singularities. In string theory, one usually finds that timelike singularities are generically much easier to "resolve" than spacelike ones (see

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however, [34] for an attempt at resolving spacelike singularities). Typically, this is because it is easier to construct specific static models which in the low energy SUGRA limit look singular, than to consider a fully dynamical set-up. Recently, there has also been renewed interest in resolving null singularities, cf. [35]–[42].

While the "singularity resolution" mechanism is model-specific, the hope would be to take advantage of the universal features. To be precise consider a scenario wherein two hitherto unrelated parent spacetimes, one with a spacelike $\mathcal{M}_s$ and other with a timelike singularity $\mathcal{M}_t$, lead to the same null singular plane wave background $\mathcal{P}$ upon taking appropriate Penrose limits. Assume furthermore that there is enough dynamical information at our disposal to resolve the timelike singularity in $\mathcal{M}_t$ to a smooth spacetime $\mathcal{M}_t^{\rm res}$. In such an event one can "resolve" the singular plane wave spacetime $\mathcal{P}$ to $\mathcal{P}^{\rm res}$, by demanding that $\mathcal{P}^{\rm res}$ be obtained from $\mathcal{M}_2^{\rm res}$ by a Penrose limit analogous to the one used to get $\mathcal{P}$ from $\mathcal{M}_t$. Running the same logic in the direction of $\mathcal{M}_s \to \mathcal{P}$, it might be possible to define a resolution of the spacelike singularity to a smooth spacetime $\mathcal{M}_s^{\rm res}$.

Whereas in a previous work [18] we asked the physical question of whether there can be black holes in pp-waves, here we have been concerned with finding the full causal structure of pp-wave spacetimes. While our findings confirm our previous results, from the string theory point of view this may appear as a somewhat more esoteric question. Causal structure is a classical notion, and our ideal point construction is cast entirely within the classical theory of general relativity. Within this setting, it is a fundamental concept, as it tells us which points in the spacetime manifold are in causal contact with which other points, and therefore the causal communication possible in the spacetime. When the curvatures are small, and especially when the curvature invariants vanish as in the present case, we generally expect this classical description to mimic what the "fundamental" objects in string theory, such as strings, see.

However, the issue is more subtle, as hinted to by e.g. the black hole information paradox. What fundamentally nonlocal objects see is not quite the same as what classical point particles see. For instance, as pointed out by [43, 44] in the context of AdS/CFT duality, an event horizon need not pose a fundamental obstacle to holographically extracting information from inside the black hole.[14] These examples suggest that objects in string theory may see more of the spacetime than allowed by classical causality. On the other hand, one would expect that communication which is allowed by causality is also allowed within string theory; in this sense, the classical causal structure of a spacetime would provide a "lower bound" on what can influence or be influenced by what.

At the same time, there are some indications that the causal structure as seen by strings might be influenced by other fields in the supergravity background, such as the $NS-NS$ two-form field. Consider the case of the Nappi-Witten model [24] as in (4.12). The metric in this solution is supported by a NS-NS field strength $H_3 = \ell\, du\wedge dz^1\wedge dz^2$. When we study the world-sheet sigma model in light-cone gauge one can make a non-local transformation on the world-sheet to cast the massive fields $\vec{z} = (z^1, z^2)$ into free fields without a mass term. From this viewpoint, it appears as though the 'effective' spacetime in terms of the

[14]For various related discussions, see e.g. [45, 46, 47, 48].

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world-sheet sigma model is flat space. However, as we have explicitly shown, the causal structure of (4.12) is quite different from the corresponding 10-dimensional Minkowski space and it in fact has a one-dimensional causal boundary.

Perhaps a more important (but murkier) use of determining the causal structure of a spacetime has to do with possible holographic duals of string theory on that background. For example, the fact that the boundary of AdS is timelike allows for a natural formulation of a dual lorentzian description; in particular, the global time in AdS spacetime is identical to the gauge theory notion of time. On a less rigorous footing is the example of the proposed dS/CFT correspondence, cf. [49], where the fact that $\mathcal{I}$ is spacelike suggests an euclidean dual. In the present case of pp-waves where we find a null one-dimensional causal boundary, we may ask what does this imply. It has been speculated [26, 15] that the one-dimensionality of the boundary suggests a dual description in terms of pure quantum mechanics.

One has to be cautious, however, in nourishing such expectations, for the following reasons. Firstly, although in AdS/CFT, the dual gauge theory is said to "live on the boundary" of AdS, there is no reason for this notion to be applicable elsewhere — even in cases where a holographic dual does exist. In particular, in the spirit of Bousso's holographic bounds, [50], the holographic screens need not correspond to the spacetime boundary. (The fact that they do for AdS may be regarded as some support for this picture.) From this standpoint, the causal structure of a spacetime may well be entirely irrelevant.

Secondly, as we mentioned at the end of section 3, the causal boundary may not be the same thing as the conformal boundary. In particular, the added "endpoints" of endless spacelike curves need not correspond to the ideal points added to future/past-endless causal curves. In fact, even the topology may differ [21]. Unfortunately, from the lessons that one gleans from the AdS/CFT correspondence, one might expect that the conformal boundary may be more important than the causal one.

It would be interesting to understand these issues better, and in particular, to find whether there is (and if so, what is) the stringy analog of the causal structure (cf., [51, 52], for some thoughts along these lines). Unfortunately, it's not even clear what we would mean by such an analog in the full string/M-theory: So far, we have been used to dealing with string theory formulated on some fixed background spacetime; however, one has to allow for the possibility that at the microscopic level spacetime itself is a derived concept, at best to be treated on equal footing with other collective field excitations. It would be very interesting to resolve these issues to enable us delve deeper into the workings of a quantum theory of gravity.

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A. Geodesic equations and properties of pp-waves

In this appendix, we will summarize some useful facts about plane waves (2.3) and pp-waves (2.2), referred to at various stages in the paper. Namely, we first derive the geodesic equations in plane waves, and then list curvature tensors and geodesic equations for the more general pp-waves. The corresponding plane wave quantities can obviously be obtained by direct substitution.

A.1 Geodesics in general plane waves

Let us consider the general plane wave metric (2.3). Explicitly writing out the summations on transverse indices $i, j = 1, \ldots, d-2$ in d-dimensional spacetime (as we will do henceforth in this section to avoid confusion with components), the general plane wave metric is

$$ds^2 = -2\, du\, dv - \sum_{i,j} f_{ij}(u)\, x^i x^j\, du^2 + \sum_i dx^i\, dx^i\,. \tag{A.1}$$

First consider the null geodesics in this spacetime. Denote by

$$p^a = \dot{u} \left(\frac{\partial}{\partial u}\right)^a + \dot{v} \left(\frac{\partial}{\partial v}\right)^a + \sum_i \dot{x}^i \left(\frac{\partial}{\partial x^i}\right)^a \tag{A.2}$$

the tangent vector to the null geodesic. Since $\left(\frac{\partial}{\partial v}\right)^a$ is a Killing field, $p_a \left(\frac{\partial}{\partial v}\right)^a = \dot{u}$ is a constant of motion, which we can set to $\dot{u} \equiv 1$, so that u acts as the affine parameter along the geodesic. The null condition then implies

$$p_a\, p^a = -2\, \dot{v} - \sum_{i,j} f_{ij}\, x^i\, x^j + \sum_i (\dot{x}^i)^2 = 0 \tag{A.3}$$

where f_{ij}, x^i, and v are implicitly functions of u, and $\dot{} \equiv \frac{d}{du}$. The Christoffel symbols for (A.1) can be easily found to be

$$\Gamma^v{}_{uu} = \frac{1}{2} \sum_{i,j} \dot{f}_{ij}\, x^i\, x^j\,, \qquad \Gamma^i{}_{uu} = \sum_j f_{ij}\, x^j = \Gamma^v{}_{ui} \tag{A.4}$$

(with all other components vanishing), so that the geodesic equations are

$$\ddot{v} + \frac{1}{2} \sum_{i,j} \dot{f}_{ij}\, x^i\, x^j + 2 \sum_{i,j} f_{ij}\, x^j\, \dot{x}^i = 0 \tag{A.5}$$

$$\ddot{x}^i + \sum_j f_{ij}\, x^j = 0 \tag{A.6}$$

However, we can use the first order constraint equation (A.3) instead of (A.5), as can be checked by integrating (A.3) and using (A.6) to obtain (A.5). Using (A.3) and (A.6) leads to further simplification, since $\dot{v}$ can be rewritten as a total derivative, so that we can solve for v in terms of x^i:

$$v = \frac{1}{2} \sum_i x^i\, \dot{x}^i + v_0 \tag{A.7}$$

where v_0 is an arbitrary integration constant which is fixed by the initial conditions.

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A.2 Overview of pp-waves

For the pp-wave metric (2.2), with $F(u,\vec{x}) \equiv F(u,x^i)$,

$$ds^2 = -2\,du\,dv - F(u,\vec{x})\,du^2 + \sum_i dx^i\,dx^i \tag{A.8}$$

the Christoffel symbols are

$$\Gamma^v{}_{uu} = \frac{1}{2}\partial_u F(u,\vec{x})\,, \qquad \Gamma^i{}_{uu} = \frac{1}{2}\partial_i F(u,\vec{x}) = \Gamma^v{}_{ui} \tag{A.9}$$

(with all other components vanishing), where $\partial_i \equiv \partial/\partial x^i$, etc. The Riemann tensor is given by

$$R_{uiuj} = \frac{1}{2}\partial_i\partial_j F(u,\vec{x}) \tag{A.10}$$

with all other components vanishing, so that the Ricci tensor has the only nontrivial component

$$R_{uu} = \frac{1}{2}\nabla_x^2 F(u,\vec{x}) \tag{A.11}$$

where ∇_x^2 is the transverse laplacian. The Ricci scalar then vanishes, $R=0$, so the Einstein tensor is identical to the Ricci tensor. For completeness, we note that the Weyl tensor is given by

$$C_{uiuj} = \frac{1}{2}\left(\partial_i\partial_j F(u,\vec{x}) - \frac{1}{d-2}\,\delta_{ij}\,\sum_k \partial_k^2 F(u,\vec{x})\right)\,. \tag{A.12}$$

Now, let us consider null geodesics in the pp-wave background. In the above coordinates, these are given by

$$\ddot{x}^i + \frac{1}{2}\partial_i F(u,\vec{x}) = 0 \tag{A.13}$$

$$\dot{v} = -\frac{1}{2}F(u,\vec{x}) + \frac{1}{2}\sum_i (\dot{x}^i)^2 \tag{A.14}$$

Note that unless $F(u,\vec{x})$ takes the plane wave form $f_{ij}(u)\,x^i x^j$, $\dot{v}$ is not a total derivative, so we can't write a general formula for v analogous to (A.7), without first solving the x-equations (A.13).

B. Null geodesics in vacuum pp-waves

We present numerical solutions to geodesic equations in vacuum pp-wave backgrounds to illustrate our point that these spacetimes are geodesically incomplete. We will deal with two main examples in the following.

Ex.1. The first example is the spacetime

$$ds^2 = -2\,du\,dv - \left[35\,z^4 - 30z^2\,(z^2+x^2+y^2) + 3\,(z^2+x^2+y^2)^2\right]\,du^2 + dx^2 + dy^2 + dz^2\,. \tag{B.1}$$

This spacetime is a vacuum pp-wave in five spacetime dimensions. It can be cast into the familiar form of (5.1), by writing the function $F(z,x,y)$ appearing in g_{uu} as $r^4 P_4(\cos\theta)$, where $P_4(x)$ is the Legendre polynomial of degree 4.

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The geodesic equations (A.13) in the background (B.1) are

$$\begin{aligned}\ddot{x} + \left(6\,x(u)^3 \; + 6\,x(u)\,y(u)^2 - 24\,x(u)\,z(u)^2\right) &= 0\\ \ddot{y} + \left(6\,y(u)^3 \; + 6\,x(u)\,y(u)^2 - 24\,y(u)\,z(u)^2\right) &= 0\\ \ddot{z} + \left(16\,z(u)^3 - 24\,z(u)\,\left(x(u)^2 + y(u)^2\right)\right) &= 0\end{aligned} \qquad \text{(B.2)}$$

We have numerically integrated the equations (B.2) and found that the geodesics typically reach infinite values of the coordinates in finite affine parameter. One such geodesic is illustrated in figure 7.

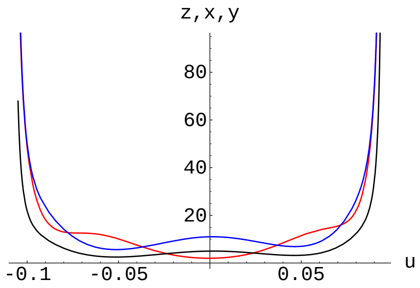

Figure 7: Geodesics $\{z(u), x(u), y(u)\}$ in the background (B.1).

Ex.2. Our next example is the pp-wave background with $F(u, r \to \infty, \Omega) \to 0$, discussed in **PP3**. Specifically, the spacetime background we consider is

$$ds^2 = -2\,du\,dv - \frac{1}{r^4}\,\left(5\cos^3\theta - 3\cos\theta\right)\,du^2 + dr^2 + r^2\,\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,. \qquad \text{(B.3)}$$

We will first take a short detour to determine the geodesics in a general 5-dimensional spacetime, and then proceed to solve them for the particular spacetime (B.3) at hand.

Since to determine the causal structure, we would like to consider radial curves, let us consider the null geodesics in the spherical coordinates. To that end, we start with the metric (5.1). If F has no angular dependence, $F(u, r, \Omega) = F(u, r)$, then we obviously have $\ddot{r} + \frac{1}{2}\partial_r F(u, r) = 0$ and $2\dot{v} = -F(u, r) + \dot{r}^2$, with $\dot{\Omega} = 0$. However, if F has some angular dependence, these no longer describe null geodesics; in other words, the geodesics cannot remain radial.

To illustrate this, let us consider a 5-dimensional spacetime, $d = 5$. For $F{=}F(u, r, \theta, \phi)$, the geodesic equations are

$$\begin{aligned}\ddot{v} + \frac{1}{2}\dot{F} + \partial_r F\,\dot{r} + \partial_\theta F\,\dot{\theta} + \partial_\phi F\,\dot{\phi} &= 0\\ \ddot{r} + \frac{1}{2}\partial_r F - r\left[\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right] &= 0\\ \ddot{\theta} + \frac{1}{2r^2}\partial_\theta F + \frac{2}{r}\,\dot{r}\,\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 &= 0\\ \ddot{\phi} + \frac{1}{2r^2\,\sin^2\theta}\partial_\phi F + \frac{2}{r}\,\dot{r}\,\dot{\phi} + 2\frac{\cos\theta}{\sin\theta}\,\dot{\theta}\,\dot{\phi} &= 0\end{aligned} \qquad \text{(B.4)}$$

As before, we can exchange the v-equation for the first order null constraint,

$$2\dot{v} = -F + \dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) \qquad \text{(B.5)}$$

as can be easily checked by integrating (B.5) and substituting in (B.4).

Specializing to the background (5.24) where $F(r, \theta) = \frac{1}{r^4}\left(5\cos^3\theta - 3\cos\theta\right)$, we find that the geodesic equations reduce to

$$\begin{aligned}\ddot{r} - \frac{2}{r(u)^5}\left(5\cos^3\theta(u) - 3\cos\theta(u)\right) - r(u)\,\dot{\theta}(u)^2 &= 0\\ \ddot{\theta} - \frac{1}{2r(u)^6}\left(15\cos^2\theta(u) - 3\right)\sin\theta(u) + \frac{2}{r(u)}\,\dot{r}\,\dot{\theta} &= 0\,.\end{aligned} \qquad \text{(B.6)}$$

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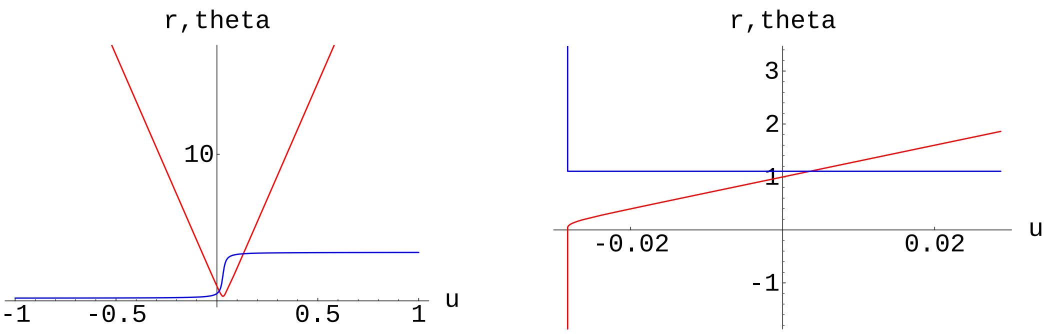

Figure 8: Numerical solutions to geodesics $\{r(u), \theta(u)\}$ for the spacetime background (5.24). The plot on the left shows geodesics which exist for the full range of the affine parameter u. The plot on the right shows geodesics which diverge at finite values of the affine parameter.

The results of numerically solving (B.6) are plotted in figure 8. As mentioned in **PP3**, there are two types of behaviour. The first plot on the left, demonstrates that there are indeed geodesics which are well behaved for all values of the affine parameter. However, from the point of view of determining the causal structure, they don't help us reach large negative values of v. This happens because $\dot{v}$ is determined in terms of $\dot{r}^2$ and $\dot{\theta}^2$, since the function $F(u, r, \theta) \sim \frac{1}{r^4} \to 0$ for sufficiently large r. This rapid approach to flat space causes the geodesics to tend towards $v \to +\infty$ rather than $v \to -\infty$. In the second plot, on the right of figure 8, we show that this spacetime is geodesically incomplete. The geodesics in this case diverge at finite values of the affine parameter.

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